LP Sparse Spike Impedance Inversion

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Introduction

Among many approaches been made to improve interpretation of post stack seismic section, a great effort has been made aimed at increasing the resolution power of interpreting section and mapping the seismic reflection data to lithology. The purpose of sparse spike impedance inversion is aimed at this goal, to obtain a high-resolution impedance profile from low resolution or band limited seismic data. The final inverted impedance possesses a broad band of spectrum, displays a blocky structure in time domain, and directly relates to lithology or petrophysical properties of rock formation.

A recorded seismic data has limited bandwidth, it usually lacks of low frequency and high frequency content. A typical seismic data has a frequency band range from 10Hz to about 80Hz. To get a broadband spectrum of impedance profile, additional information is needed. That information generally comes from well log data available in the survey area, and our a priori knowledge of local geology. The final inverted impedance section will contain more information than the original seismic data, and therefore more suitable for the interpretation work.

Inversion Method

The goal of impedance inversion is to find a model that is reflection coefficients \( r(t) \) given that seismic data \( x(t) \) and source wavelet \( w(t) \) as input, and then calculate impedance \( \xi(t) \) from the model. The relationship between data \( d=(x_1, x_2, \ldots) \), model \( m=(r_1, r_2, \ldots) \) and noise \( n \) is described as the following:

\[ Lm + n = d, \]

where \( L \) is the operator that links the model to the observation. With the known observation \( d \), the model can be defined with the probability \( p(m|d) \), the posterior probability of the model conditional on the observation. It can be described with the Bayes’ formula:

\[
p(m | d) = \frac{p(d | m)p(m)}{p(d)},
\]

where \( p(m) \) and \( p(d) \) describes our prior knowledge on model and data respectively. To build a model based on the probability \( p(m|d) \), we may seek a MAP solution that maximize a posterior probability \( p(m|d) \). Our objective function for minimization can be chosen as:
\[ J = -\log p(m|d) = -\log p(d|m) - \log p(m). \]

The term \( \log p(d) \) is ignored from \( J \) since it is a constant. Our prior knowledge of the model is generally given as a global constraint \( S(m) \), and its probability distribution that gives the maximum entropy and subject to the constraint takes the following form (Sacchi and Ulrych),

\[ p(m) = A e^{-S(m)}, \]

where \( A \) is a normalization constant. Assume a generalized Gaussian modelisation and observational error, we have probability \( p(d|m) \) described as a function of discrepancy between the model and the observation,

\[ p(d|m) = \frac{p^{1-1/p}}{2\sigma_p \Gamma(1/p)} \exp\left(-\frac{1}{p}\frac{|d - Lm|^p}{\sigma_p^p}\right). \]

Choose \( p \) equals 1 or 2, we will have a solution of \( l_1 \)-norm or \( l_2 \)-norm. Maximize the objective function \( J \), we obtain a solution that gives least error between the model and the observation, and meanwhile honors our prior knowledge of the model.

**\( l_1 \)-norm Solution**

From our general knowledge of geology and the well log observations, we know that the earth displays a layered structure, and its velocity or impedance has a trend of increase with depth. However the impedance obtained by direct inverting seismic trace shows no resemblance to what we have expected. Such a distorted picture is due to insufficient information contained in seismic data that forbid us from extracting the complete information on impedance structure.

We may divide the frequency spectrum of an earth model structure into three distinct regions as indicated in figure 1. The frequency spectrum of an input seismic data is generally only overlap with the middle frequency band, the seismic band. Lacks of low and high frequency content in seismic data result in non-uniqueness in the impedance inversion. The seismic observation can only provide the constraint to the model on seismic wave band. Any values beyond the seismic frequency band will generate a model that is compatible with seismic data, but it may not necessarily be the right one.

The challenge of impedance inversion is to create an impedance model that honors seismic data for its frequency band and meanwhile incorporate as much our prior information on the earth model to reduce such non-uniqueness. We consider an earth model consists of many layers, and its reflectivity function has the following mathematical form (Oldenburg et al. 1983),
\[ r(t) = \sum_{j=1}^{NL} r_j \delta(t - \tau_j). \]

Where \( NL \) is the total number of layers in the model, and \( r_j \) is the corresponding reflection coefficients to each layer. Since \( NL \) is a much smaller number than the \( N \), the number of samples in recorded seismic, it greatly reduces the degree of freedom of the model, therefore reduces the non-uniqueness.

Among all the compatible models that honor seismic observation, we carefully choose our solution that honors well log data for the low frequency band, and honors the assumption of layered geology for high frequency band. The former is the requirement of low frequency constraint; and the latter is the requirement of sparse reflectivity, which assumes that the earth model consists of a number of homogeneous layers, and we choose the one that yield least number of layers.

![Power Spectrum](image)

Figure 1. The frequency spectrum of the earth model versus seismic data.

To find a \( l_1 \)-norm solution, we take the constraint of the form \( S(m) = \sum_{n=0}^{N-1} |r_n| \). The objective function we are going to minimize become

\[ J(m) = \sum_{n=0}^{N-1} |r_n| + |Lm - d|. \]

The advantage of using \( l_1 \)-norm is that the minimization with \( l_1 \)-norm tends to give a robust sparse solution than the minimization with \( l_2 \)-norm. When the data contains many outliers, we may still reach a reasonable solution with \( l_1 \)-norm solution but fail with \( l_2 \)-norm. Linear programming (LP) approach suggested by Oldenburg et al. (Oldenburg et al. 1983; Levy and Fullagar 1981) provides an efficient way to solve the above system. It recovers an impedance model with sparse reflectivity, yields the fewest number nonzero reflection coefficients. Hence a desired isolated delta function model, or equivalently, an
earth model having the smallest number layers could be recovered. Such algorithm would pick out the major features in the acoustic impedance structure; recover an earth model having the smallest number of layers.

To adopt a convolutional model, we may assume that after seismic data is properly processed it may be modeled by the convolution between a wavelet and a reflection coefficient series:

\[ x(t) = r(t) \ast w(t), \]

where \( x(t) \) is seismic data, \( w(t) \) is wavelet and \( r(t) \) is reflection coefficient. When the changes of acoustic impedance defined as \( \xi = \rho v \) is small, the reflection coefficients relates acoustic impedance by

\[ r(t) = \frac{1}{2} \frac{d[\ln \xi(t)]}{dt}. \]

Therefore the impedance may be obtained through integration:

\[ \xi(t) = \xi(0) \exp \left[ 2 \int_0^t r(\tau) d\tau \right]. \]

From the above relation, we may see that the sparseness in \( r(t) \) may result in a blocky structure in \( \xi(t) \).

For the frequency within seismic frequency band, we may write following equations as constraints

\[ X_j = W_j \sum_{k=0}^{N-1} r_k e^{-2\pi j k/ N}, \]

where \( X_j \) and \( W_j \) are Fourier components of seismic data and wavelet. They are given as

\[ X_j = \sum_{k=0}^{N-1} x_k e^{-2\pi j k/ N}, \]

\[ W_j = \sum_{k=0}^{N-1} w_k e^{-2\pi j k/ N}. \]

For the low frequency below seismic frequency band, we may extract the low frequency component of the reflectivity function from the input model constructed from well log data
For the high frequency above seismic frequency band, we may impose an inequality constraint to limit its power

\[
\sum_{k=0}^{N-1} r_k e^{-2\pi j k/N} \leq \sigma .
\]

The parameter \( \sigma \) describes the high frequency contents in the inverted impedance. Therefore it describes the sparseness of the reflectivity function.

These equations provide a linear relationship between the Fourier transform of the reflectivity function, the wavelet and the reflection coefficients. A linear programming algorithm can be used to minimizing the objective function subject to the above constraint equations.

The detailed description of the algorithm of linear programming may be found in Barrodale and Roberts's (1978) paper. Their algorithm minimizes a linear objective function with \( l_1 \)-norm subject to given linear constraints in equality or in inequality form. The algorithm modified traditional simplex approach to increase the efficiency of computation and convergence.

To improve the computational efficiency and stability, we divide a seismic trace to be inverted with a number of overlapping windows. The advantage of using small windows is two fold. First, the computational cost in solving a group of linear equations by minimizing the \( l_1 \)-norm of residual error increases tremendously as number of unknowns or equations increase. Dividing a seismic trace into a number of small segments and work on each individually make the computational cost increase linearly with the increase of the equations. Second, the LP inversion scheme described above is formulated in frequency domain. A disadvantage of working in frequency domain is that a local noise tends to smear out to the entire trace. Using small windows we may limit us to a local region in a trace, and therefore reduce the effect of error smearing.

**Examples**

In this section we will use some synthetic and real data examples to demonstrate the LP impedance inversion discussed above and the effects of user controlled parameters on the inversion results.

Figure 2 displays a synthetic impedance log with which we generate a synthetic seismic trace with a band-limited wavelet. We then try to recover this impedance log from the generated seismic data.
To start impedance inversion, we first create an initial impedance model shown as the curve displayed at the bottom of figure 3. The red curve in the middle is the synthetic trace calculated from the initial model, and the black trace is the input seismic data. The curve at the top of the figure is the residual error between the synthetic and the input data.
Figure 4. The model displayed in red at the bottom of the figure is the inverted impedance model. The synthetic trace calculated with the inverted model matches with the input seismic trace perfectly. It yields a very small residual error displayed at the top of the figure.

As we could see from figure 3 the initial model is a smooth log. However, after LP impedance inversion, a blocky model is recovered as shown in figure 4. The synthetic trace calculated with the inverted model matches with the input seismic trace perfectly. This implies that the information contained in the seismic trace is mostly recovered. In addition to that, the inverted model has the low frequency trend of the initial model, and a blocky structure. Compare with the true model displayed in figure 2, we could see the resemblance between the inverted model and the true model. The major characteristics of the true model are faithfully recovered.

To test the effect of the initial model on the inversion result, we go through the same exercise with a blockier initial model as displayed in figure 5. Again we could see the major characteristics of the true model are recovered (figure 6). The discrepancy between the true model and the inverted model is mainly come from the low frequency discrepancy of the initial model from the true model. The inversion method heavily depends on the initial model for the recovery of low frequency components. Poor initial model in low frequency may have more impact on the inversion result.

Figure 7 displays a real data example. The initial model displayed at the bottom of the figure is extracted from the nearly well log data. The seismic trace displayed in the middle of the figure as a black curve is extracted from a poststack seismic volume. The figure 8 shows the inversion result. It is shown that the synthetic seismic trace calculated based on the inverted model has a good match with the real data. However, the inverted model has a much wider frequency spectrum than the original input seismic data.
The above inversion exercise is done with the parameter \textit{sparseness} set at 100%. This will give a most sparse spike model solution. By reducing the parameter sparseness to 50%, we will obtain a less sparse spike model solution as displayed in figure 9. Further reduce the sparseness to 20%, we obtain a result similar to that from the band limited inversion (figure 10). While the sparseness of the model changes, the match between the synthetic and real seismic data always retains. This is because the information contained in the seismic wave band is always honored during the course of inversion.

\textit{Window length} is another user input parameter in the LP sparse spike inversion menu. It is the number of samples of the overlapping windows, or the operator length used in the inversion. The above example is calculated with the window length 256 samples. To see the effect of this parameter on the inversion result, we apply different window length to the data. Figure 11 displays the inversion result using 128 samples as the window length. Figure 12 displays the inversion result using 64 samples as the window length. The major characteristics of the inverted models are agreed with the result displayed in figure 8. Nevertheless, short window length seems to show more discrepancy between synthetic and real seismic data. The reason for this lager discrepancy is due to truncation error in the low frequency components. Shorter window length in time domain gives coarser sample rate at the frequency domain, and therefore affects the accuracy of the \textit{maximum constraint frequency}, another user input parameter in the LP sparse spike inversion menu.

Like band limited inversion, LP sparse spike inversion depends on the initial model to extract low frequency components. The parameter \textit{maximum constraint frequency} sets the threshold for the maximum frequency to be extracted from the model. The higher is this parameter, the more close of the inverted model will be to follow the trend of the initial model. In above exercises we use maximum constraint frequency 5Hz. To demonstrate the effect of this parameter on the inversion result, we set it to 10Hz, and rest of the parameters is the same as used for the figure 8. The result of the inversion is displayed in figure 13. Compare with the figure 8, we can see that the inverted result in figure 13 follows the trend of the initial model more closely than that in the figure 8. Since we ignored more low frequency components in seismic data, it causes larger discrepancy between the synthetic and the input seismic data.

We apply the inversion to the entire 3D data volume with default parameters, which is sparseness equals 100%, maximum constraint frequency equal 10Hz and window length equals 128 samples. One of inline sections of the inversion result is displayed in figure 14. The inverted impedance model is displayed in both color scale and wiggle traces. As can be seen from the figure, the impedance model shows a blocky characteristics and good continuity in lateral direction.
Figure 5. The bottom curve is the initial model. The black trace in the middle is the input seismic and the red trace is the synthetic trace calculated with the initial model.

Figure 6. The model displayed in red at the bottom of the figure is the inverted impedance model. The synthetic trace calculated with the inverted model matches with the input seismic trace perfectly. It yields a very small residual error displayed at the top of the figure.
Figure 7 shows a real data example. The initial model at the bottom is extracted from nearby well log data. The input seismic trace displayed as a black curve in the middle is extracted from a poststack seismic volume. The red trace is a synthetic trace calculated from the initial model.

Figure 8 shows the inversion result. The inverted model shows blocky characteristics. The synthetic seismic trace calculated with this model matches well with the real seismic data.
Figure 9 shows the effect of the parameter *sparseness* on the inversion result. Setting the parameter to 50%, we obtain a less blocky inversion result.

Figure 10 shows the inversion result with the parameter *sparseness* equals 20%. While a less sparse spike model we get, the matches between the synthetic and the real seismic data retains.
Figure 11 displays the inversion result using the parameter *window length* 128 samples.

Figure 12 displays the inversion result using the parameter *window length* 64 samples.
Figure 13 displays the inversion result with the *maximum constraint frequency* at 10Hz.

Figure 14 displays one of inline sections of the inverted impedance model. Both color scale and wiggle traces display the impedance strength.

**References**


**More references**


