A little math, a little physics[1]

1 Physical quantities

1.1 Units - SI

While there are dimensionless or unit-free quantities that are important in physical and geophysical measurements, most quantities are measured or scaled according to the unit system we have chosen to use. We shall try to stick, rigorously, to the **SI**, **Système Internationale**[2], unit system. In principle, we in Canada and in our teaching at McGill have been obliged to use (only) SI in our courses since 1976. In the US, SI is used only in science[3]. Unfortunately, many geophysical and physical measurements are still quoted in archaic unit systems, especially the **CGS-EMU** and **CGS-ESU** and the variant **Gaussian** system of units[4]. All the old physicists and geophysicists who were educated to use these unit systems aren't dead yet! One issue that complicates matters is that the formalization of physics, itself, looks different in each of these three unit systems. Geophysics, except with respect to seismic exploration in North America and, then, only until the mid-1970s, has not used the British Engineering Unit system or the American adaptation of it. You shouldn't see miles, gallons, feet, inches, poundals, pounds, etc.. in this course. Now, from time-to-time, I might, though. try to relate to these because the colloquial unit system used by our neighbo**u**r to the south still employs these units of measure in everyday use and as some of you come to McGill with this experience, I'll try to help with conversions of these units into those used in science.

You might not know that the "inch" was only standardized in the 1960s. The pressure to standardize was driven by the need for extremely precise machine work and design for the US space program. Until then, the US used the American inch which was exactly 1/39.37][5] of the standard "metre" held in France. The British inch was 1/36 of the length of the standard "yard" held in Britain. The Canadian "inch" was a little smaller (by 2.00000400001 parts in one million) than the American inch and little larger than the British inch: it was (is) 2.54] centimeters where 1 centimeter is 1/100 of the standard metre held in Paris. This is the inch that was chosen for the high-precision machine work for the space program and NASA. NASA and the US space program is, now probably, the only scientific endeavour that still employs the adapted BEU system at least in it engineering.

[1] ***... and some geography***
[2] Internet links are shown in blue:The NIST Reference for SI
[3] History of the Metric System in the US
[4] The two CGS unit systems
[5] I indicate an exact number by termination with a closing square bracket... on use of numbers "Measured" numbers follow the normal rule for uncertainty as indicated by significant figures

Système Internationale

Quantity	SI unit	Unit name
distance	m	metre
mass	kg	kilogram
time	s	second
current	A	ampère

In principle, we can scale all possible physical quantities in terms of these 4 units. Practically, for convenience, we describe myriad derived units relevant to particular measures. One such unit, for example, is the V (volt) which is derived as

$$1V = 1\frac{kg \cdot m^2}{A \cdot s^3} \quad .$$

1.1.1 Definitions

- **The metre:** $1m$ is the distance light travels in a vacuum in 1/299792458] of a second. Thus the metre depends on the definition of the "second". The "speed of light in vacuo" is defined ***exactly*** as a natural constant: $c = 299\,792\,458]\, m \cdot s^{-1}$.

- **The kilogram:** $1kg$ is now determined as the mass of the *International Prototype Kilogram* which is made of a platinum-iridium alloy and which is stored in Sèvres, France. The IPK is known to be evaporating and so decreasing in mass. It is intended to redefine the kilogram in terms of something more stable and standard but no decision has yet been made as to how. One proposal is to define it as $4.5946723] \times 10^7$ Planck masses; another suggestion is define it in terms of its $E = mc^2$ energy equivalence as $89875517873681764]\, J$. That is, then, an energy measure would become the mass-standard unit in the SI system. Presently, J, the "joule" is the derived unit:

  $$1J = 1kg \cdot m^2 \cdot s^{-2}.$$

  The International Committee for Weights and Measures (CIPM) has proposed revised formal definitions of all the SI base units[6]. These are to be considered by the 26th General Conference on Weights and Measures, scheduled for the autumn of 2018. In their currently preferred redefinition, the kilogram becomes defined as that mass that corresponds to the ***exact*** definition of "Planck's constant" as: $h = 6.62606] \times 10^{-34} J \cdot s$. The kilogram thus becomes a derived unit dependent upon the essential definitions of the metre and second and a fundamental physical relationship to a mass.

[6] Redefining SI base units

- **The second:** Currently, $1s$ is the duration of 9,192,631,770] periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom at rest and at $0K$. This is equivalent to 1/86400 of the mean solar day. The immediate solar day depends on the Earth's rotation period which is changeable and generally lengthening. The ***mean solar day*** was defined as the average length of day between January 0, 1900 and December 32, 1905.[7]

  By the way, the common geographical coordinate system was defined to accord with its north pole penetrating the Earth's surface shell at a point which represented the mean axis of rotation of the Earth during this very same period.[8] The axis of rotation of the Earth is displaced, now, by about $19m$ from the north geographical pole... or better, the geographical coordinate system inscribed on the Earth has slipped $19m$ from the rotation axis. Montreal's position relative to the rotation axis is now about $18m$ closer (north) than it was when the geographical coordinate frame was fixed. Also, relative to the rotation axis, the geographical 0-meridian has slipped about about 0.17" to the west. At Montreal's latitude, that corresponds to about 4.7 m towards us. The Earth is a changeable and dynamic body! Geophysics! We must acknowledge as well that the shell (crust, lithosphere) of the Earth is in constant vertical and lateral adjustment as well and this means that places on the Earth are locally moving through the geographical cooridinate system. Geology!

- **The ampère:** $1A$ is currently defined operationally as that "constant current which will produce an attractive force of $2 \times 10^{-7} kg \cdot m \cdot s^{-2}$ (newtons) per metre of length between two straight, parallel conductors of infinite length and negligible circular cross section placed one metre apart in a vacuum". This unit, like the kilogram is to be redefined in somewhat more practical or physically fundamental terms in 2018. The CIPM proposal defines the charge on one electron as $e = 1.60217 \times 10^{-19} A \cdot s$ ***exactly*** and hence the derived definition of the ampere.

- **Other basic physical units** The definitions of the **kelvin**, the **mole** and the **candela** will be reviewed in 2018.

Units are extremely important as almost every physical quantity is measured in terms of these units. We can only compare quantities if we can compare in terms of fixed units.

[7] Excess LOD
[8] Pole location in geographical coordinates

2 Tensors and their special cases

One might say that all physical quantities are described in terms of **tensors**. Most physical quantities that we commonly measure, though, are determined in the form of special cases of tensors, namely **scalars** and **vectors**. Before dwelling on the character of tensors, we should understand another construction of mathematical and physical convenience: a coordinate system.

2.1 Coordinate systems

We establish coordinate systems to determine where we are or some point or event is in space and time. Usually, in simple physical and geophysical problems, we try to employ the simplest coordinate system that can locate us and the objects and phenomena that we are studying most easily. On the surface of the near-spherical Earth, we locate ourselves by latitude and longitude referenced to that geographical coordinate system established in 1905. That Earth is not perfectly spherical – it has highlands and lowlands and is flattened by about 1 part in 300 along the rotation axis – actually seriously complicates locating ourselves on the surface. The very most mathematically intense field of geophysics, geodesy, is concerned with this serious location. The most elaborated mathematics of differential geometry is required to locate... Einstein learned differential geometry from his first wife during the 10 years that he was generalizing his 1905 theory of "Special Relativity". That so-called "General Relativity", now called "Gravitation Theory", is thought to be so inaccessible to many people is because it requires the differential geometry and tensor calculus in order to do anything with it. You won't see any differential geometry or serious tensor calculus in this course! Whew!

We have a sense that space is 3-dimensional and that placement of a point in space can be easily described in terms of 3 coordinates that we attach to our 3-space. Space, though, has a geometric character and does not necessarily nicely match to our chosen coordinate system for describing places in that space. Space is real; coordinate systems are for us to choose arbitrarily. We can describe the geometry of space in terms of how it locally matches to any arbitrary coordinate system we choose. The coordinate system belongs to us according to our convenience; space belongs to nature and is "real". Below, I shall show you how it is that we describe the geometry of a region around a point in space in terms of the "**metric tensor**". The concept is quite simple, I promise.

How far is it from here to there? The displacement from here to there is described by a **vector** quantity composed of a distance and a direction. If we choose to originate our coordinate system (Let's use a simple **Cartesian** coordinate system and presume that the geometry of space is simply **Euclidian**.) right here and then cast out an orthogonal $x-y-z$ coordinate system from here, the vector place of "there" is simply given by its particular x, y and z placements in our coordinate system. Here to there is a vector displacement. Note that our coordinate system is chosen to be scaled in distance units – say metres. If we, "here" sit at the coordinate origin ($x = X_h = 0$, $y = Y_h = 0$, $z = Z_h = 0$), then "there" is located at $x = X_t$, $y = Y_t$ and $z = Z_t$. The subscripts h and t describe "here" and "there". The vector place of "there" is (X_t, Y_t, Z_t) and of here $(0, 0, 0)$ as I have located us here at the coordinate origin.

How far is it from "here" to "there"? That question is not entirely trivial! But, as I noted above that our space was Euclidian (sometimes called "flat".) and therefore, a wonderful theorem due to **Pythagoras** gives us the answer. The distance, say ℓ_t from here to there is simply obtained as

$$\ell_t^2 = X_t^2 + Y_t^2 + Z_t^2$$

or

$$\ell_t = (X_t^2 + Y_t^2 + Z_t^2)^{1/2} \quad .$$

We can recast this Pythagorean formula into a rather nice linear-algebraic form:

$$\ell_t^2 = \begin{bmatrix} X_t & Y_t & Z_t \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} .$$

We premultiply the **identity** matrix into the column vector and premultiply that result by the row vector of the X_t, Y_t, Z_t coordinate locations of "there".

Here we have a vector described in two forms, row and column and a 3×3 matrix. Each element in each of the vectors relates to **one** of our coordinates; each element in the matrix relates to **two** of our coordinates, in pairs. For this matrix, only those elements that relate to the X-row and X-column pair, the Y-row and Y-column pair and the Z-row and Z-column pair have the non-zero value, 1. This matrix is actually a real physical quantity: it is in this simple case, the **metric tensor** of Euclidean space described through Cartesian coordinates. Our linear-algebraic form is more than just vector-matrix-vector multiplication; it is a tensor-algebraic multiplication. The metric tensor, here, "operates" mathematically, just like an ordinary matrix. It is though a real tensor in that it describes and measures a real physical quantity; it is not just a mathematical tool. The real physical quantity it describes is the real Euclidean geometry of our space.

The metric tensor is a 2-rank (not to be confused with matrix rank) tensor. That is, each element is determined by 2 coordinates. Tensors can be of any rank. The vectors, whether row or column, have elements that are determined by only 1 coordinate. Vectors are 1-rank tensors. In continuum mechanics, elasticity theory and seismology, we encounter an important 4-rank tensor, the **Elastic Coefficients tensor**[9]

[9] Elastic coefficients tensor

that describes the simplest of elasticity conditions for linear, homogeneous materials. Rocks are not! In Gravitation Theory (colloquially General Relativity) the **Riemann curvature tensor**[10] that describes the local curvature of space-time is 4-rank.

So far, we have come upon vectors and tensors. What of scalars? (We may regard them as 0-rank tensors.) I am sure that many of you know that some physical quantities have no variation or relationship to where they exist in a coordinate system. Distance (e.g., ℓ_t above) has no coordinate dependence; displacement does! Mass, whether here or there, measures locally to same quantity. Except for evaporation, our IPK kilogram is $1kg$ here or there. This quantity, mass, is a scalar quantity; it doesn't depend on coordinates at all. **But** the weight of the mass is a vector quantity! How? Weight is a force measure, typically on the Earth's surface the force of the mass under gravitational acceleration. Suppose we orient our coordinate system so that "down" is in the negative z-direction and along that direction the gravitational acceleration is $9.8m \cdot s^{-2}$ decribed by the vector $\vec{\boldsymbol{a_g}}$ **:** $[0 \quad 0 \quad -9.8]\ m \cdot s^{-2}$, then the vector weight $\vec{\boldsymbol{w}}$ **:** $[w_x \quad w_y \quad w_z]$ of a mass, $\boldsymbol{m}\ \ kg$, is simply

$$[w_x \quad w_y \quad w_z] = \boldsymbol{m} \quad [0 \quad 0 \quad \text{-9.8}] \qquad [kg \cdot m \cdot s^{-2}]$$

or in traditional vector notation,

$$\vec{\boldsymbol{w}} = \boldsymbol{m}\,\vec{\boldsymbol{a_g}}$$

with units $[kg \cdot m \cdot s^{-2}]$. Note that, properly, all elements of a vector or of a tensor carry the same physical units.

We can extend this traditional vector notation to the tensor equation for the distance above that was described in vector-matrix form:

$$\ell_{\mathbf{t}}^{\mathbf{2}} = \vec{\mathbf{X}}_{\mathbf{t}}^{\mathbf{T}} \cdot \overline{\overline{\boldsymbol{\Delta}}} \cdot \vec{\mathbf{X}}_{\mathbf{t}} \quad [m^2].$$

This is a variation of the so-called **dyadic** tensor notation. Normally I follow another notation: **indicial notation**; it renders tensor math and calculus that is more "useful" and more easily transformable into computer codes.

2.2 Mathematical operations on tensors...

Scalars add, subtract, multiply, divide just like ordinary numbers with a couple of provisions. One can only add or subtract scalar values that carry the same units. One can multiply or divide scalars but one must also multiply or divide the units they carry.

[10] Riemann curvature tensor

Vectors add and subtract, element by element, just like scalars. Vector multiplication is somewhat more complicated, though. The **vector dot product** describes one form of vector multiplication:

$$\vec{\mathbf{Z}} = \vec{\mathbf{X}} \cdot \vec{\mathbf{Y}}$$

where

$$\vec{\mathbf{X}} : [X_x \quad X_y \quad X_z], \; \vec{\mathbf{Y}} : [Y_x \quad Y_y \quad Y_z]$$

and

$$\vec{\mathbf{Z}} : [Z_x \quad Z_y \quad Z_z] = [X_x Y_x \quad X_y Y_y \quad X_z Y_z].$$

The other form of vector multiplication is the **vector cross product**:

$$\vec{\mathbf{Z}} = \vec{\mathbf{X}} \times \vec{\mathbf{Y}}$$

where

$$\vec{\mathbf{Z}} : [X_y Y_z - X_z Y_y \quad X_z Y_x - X_x Y_z \quad X_x Y_y - X_y Y_x].$$

Those of you who have taken a course in *Electromagnetic Waves and Fields* might know of an important physical quantity that is described as the vector cross product of the electric and magnetic field vectors of an electromagnetic field or wave, the **Poynting vector**:

$$\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}}.$$

This vector describes the energy flux (per unit area along its propagation direction with units $[J \cdot m^{-2} \cdot s^{-1}]$) of an electromagnetic wave – a light beam!

2.3 ... and a little linear algebra

Multiplication of 2-rank tensors (as far as we shall go, here) is again a little more complex.

- To multiply a 2-rank tensor by a scalar is quite simple: every element of the tensor is multiplied by the scalar value and its units.
- We have already seen above (concerning the Pythagorean theorem) how one multiplies a 2-rank tensor into a vector through a simple matrix-vector computation:

$$\vec{\mathbf{W}} = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} = \begin{bmatrix} \Delta_{xx} & \Delta_{xy} & \Delta_{xz} \\ \Delta_{yx} & \Delta_{yy} & \Delta_{yz} \\ \Delta_{zx} & \Delta_{zy} & \Delta_{zz} \end{bmatrix} \bullet \begin{bmatrix} Y_z \\ Y_y \\ Y_z \end{bmatrix}.$$

- We can also premultiply the vector $\vec{\mathbf{W}}$ by the transpose of the vector $\vec{\mathbf{Y}}$ as:

$$s^2 = \begin{bmatrix} Y_x & Y_y & Y_z \end{bmatrix} \bullet \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix}.$$

  to obtain a scalar value in result!

  You might assign $\Delta_{xx} = \Delta_{yy} = \Delta_{zz} = 1$ and the other 6 $\Delta_{??} = 0$ to see that s^2 becomes the squared Pythagorean length from the coordinate origin to the vector location in our 3-space $\vec{\mathbf{Y}} : [Y_x \quad Y_y \quad Y_z]$.

- We could just as well have premultiplied the transpose $\vec{\mathbf{Y}}^{\mathbf{T}}$ into the matrix $\overline{\overline{\mathbf{\Delta}}}$ and then multiplied that resulting row vector into the column vector $\vec{\mathbf{Y}}$ to come to the same result.

3 Tensor-vector calculus; Div, Grad, Curl and all that...

For the purposes of this course, we shall live in a 3-space which is characterized by an independent time measure. That is, we shall not describe any of our geophysics in terms of the 4-space-time structure that Einstein introduced with his *Special Relativity* and which he later generalized into *Gravitation Theory* through the inclusion of the space-time distortions determined by mass. For most geophysics, we can ignore space-time dilations due to speed and mass. We shall keep time independent of space in the *Newtonian* way.

Still, that we live in a 3-space and since many geophysical and geological conditions and processes vary according to place and time, we are stuck with their description in terms of multi-variate calculus, vectors, tensors and **partial differential equations**. In explaining, though, many of the geophysical processes such as, for example, thermal diffusion, we shall suppress 2 of our 3 spatial dimensions and so allow ourselves to come to some understanding of the geophysical phenomena in terms of simple **ordinary differential equations**. While everyone in our programs is expected to have taken *Advanced Calculus* or to be now taking it concurrently, I shall review some of that that you should have absorbed from that course. Unfortunately, at McGill and commonly elsewhere, mathematics is taught in *"manipulation"* without the understanding of why one might engage in manipulations. I address the why of manipulations for the moment because over my 43 years at McGill I have recognized that most students know how to do math; very few students know why or when to employ a mathematical tool. I don't care a damn that you can manipulate; I want you to know what you are doing and why you are doing it.

3.1 The gradient

I presume that everyone here has taken an introductory course in calculus and that everyone can differentiate a function in one variable. For example, given a function as simple as $\boldsymbol{f(x) = mx + b}$, a general line, I expect that everyone can differentiate that function to obtain the slope of the line. By the way, that slope is $\boldsymbol{m}$.

Now what if that function were dependent on two coordinates, say x and y as $\boldsymbol{f(x, y) = mx + ny + b}$? Then how does one find the equivalent slope? The mathematical tool we use is the **gradient operator**:

$$\boldsymbol{\nabla(f(x,y)) = [\partial_x f(x,y)_x \quad \partial_y f(x,y)_y] = [m \quad n].}$$

This is a vector quantity describing an x-component of the slope, $\boldsymbol{m}$ and a y-component of the slope, $\boldsymbol{n}$.

Above, the forms $\boldsymbol{\partial_x(\,)}$ and $\boldsymbol{\partial_y(\,)}$ represent the **partial derivative** of whatever is within the $\boldsymbol{(\,)}$ bracket with respect to the subscript variable. Sometimes we write this more carefully, as for example,

$$\boldsymbol{\partial_x(\,) = \frac{\partial}{\partial x}(\,).}$$

For our purposes, presuming that the natural phenomena we are trying to describe with our mathematics fulfill all the necessary existence conditions,

$$\boldsymbol{\frac{\partial}{\partial x}(\,) = \frac{d}{dx}(\,)|_{y,z fixed}.}$$

3.2 Divergence

Consider a vector field $\boldsymbol{\vec{B}(x, y, z) : [B_x(x, y, z) \quad B_y(x, y, z) \quad B_z(x, y, z)]}$ as sparsely diagrammed in Figure 1.

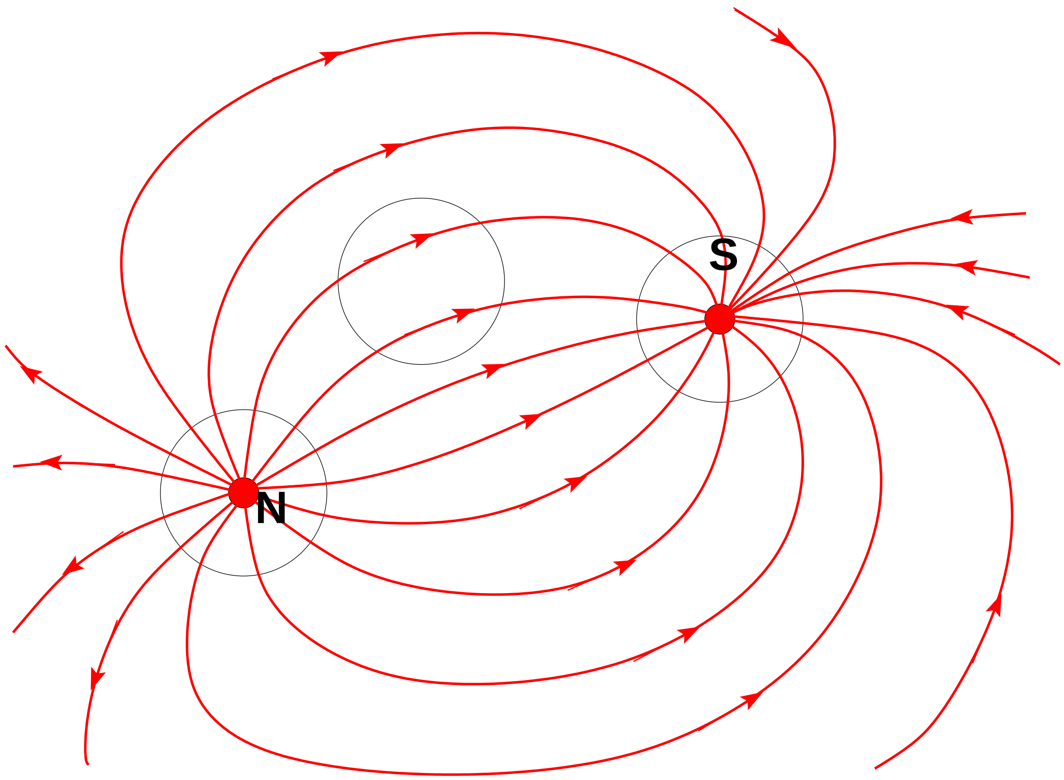

Figure 1. A vector field: source and sink.

The field of vectors – you might consider them to be magnetic field lines where the vector along the line represents the magnitude and direction of the local *magnetic induction* – issue from **source** $\boldsymbol{N}$ to be consumed at **sink** $\boldsymbol{S}$. At point $\boldsymbol{N}$, or more properly over some (impossibly) infinitesmal volume enclosing $\boldsymbol{N}$, the **divergence** of the vector field is determined as:

$$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{\partial_x B_x} + \boldsymbol{\partial_y B_y} + \boldsymbol{\partial_z B_z} > \boldsymbol{0}$$

because this is a *source* of the vector field, a *north-seeking magnetic pole.*

You might note that the divergence of a vector field is the sum of scalar differential elements and is, therefore, scalar. It may have units, of course, but it has no directional characteristics.

At point $\boldsymbol{S}$,

$$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) < \boldsymbol{0},$$

a *sink.*

In this description, I have actually misled you, not concerning the mathematics of divergence as related to sources and sinks of the vector field but concerning the very character of $\boldsymbol{\nabla} \cdot \vec{\boldsymbol{B}}$. *Gauss' Law* determines that there is no volume, no matter how impossibly small, which can isolate a source or sink of the field of magnetic induction. Everywhere $\boldsymbol{\nabla} \cdot \vec{\boldsymbol{B}} \equiv \mathbf{0},$ everywhere!

3.3 Curl

It might not surprise you that the **curl** of a vector field, $\boldsymbol{\nabla} \times \vec{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ describes just that: how the vector field curls and twists.

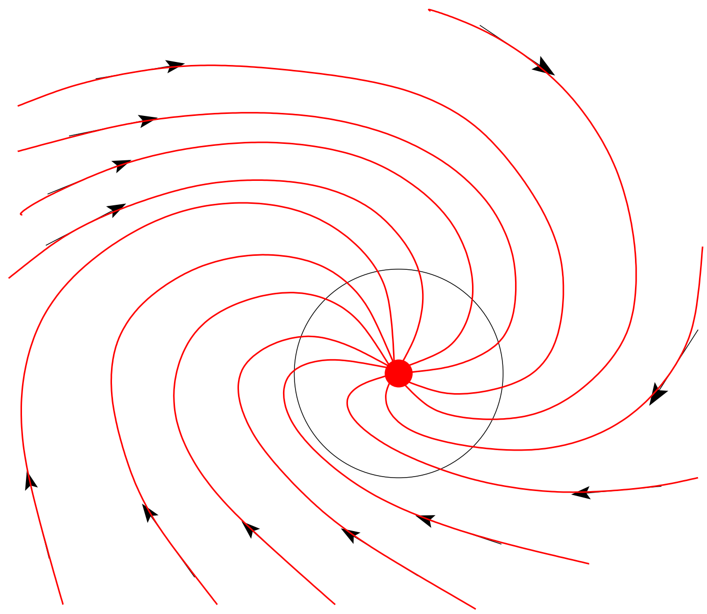

Figure 2. A vector field: vorticity.

To describe curls and twists (properly **rotation** or **vorticity**) of a vector field does require direction: a twist could be clockwise, for example, or counter-clockwise, and the the axis of this rotation could be oriented in any possible direction.

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) : [(\boldsymbol{\partial_z B_y} - \boldsymbol{\partial_y B_z}) \quad (\boldsymbol{\partial_x B_z} - \boldsymbol{\partial_z B_x}) \quad (\boldsymbol{\partial_y B_x} - \boldsymbol{\partial_x B_y})].$$

$\boldsymbol{\nabla} \times \vec{\boldsymbol{B}}$ is a vector and if, again, $\vec{\boldsymbol{B}}$ represents the vector field of magnetic induction,

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{B}} = \frac{\mathbf{1}}{\boldsymbol{\mu}}(\vec{\boldsymbol{J}} + \boldsymbol{\partial_t} \vec{\boldsymbol{D}}).$$

This is one of *Maxwell's equations*; it describes how the field of magnetic induction twists according to the local passage of a current flux density, $\vec{\boldsymbol{J}}$, and the temporal variation of the *electric displacement*, $\vec{\boldsymbol{D}}$. For example, if there were no local variations in $\vec{\boldsymbol{D}}$ and a flux of solar-wind protons were to pass by, the twisting of the magnetic induction due to that flux is described by $\boldsymbol{\nabla} \times \vec{\boldsymbol{B}}$. You should note that $\boldsymbol{\nabla} \times \vec{\boldsymbol{B}}$ is differential and one would have to integrate all the contibuting differential elements to find the vector field $\vec{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. You should also note that each term in the equation for $\boldsymbol{\nabla} \times \vec{\boldsymbol{B}}$, above, is a vector. In a tensor equation (This is an equation in a tensor of rank-1.) every term must be of the same tensor rank.

3.4 Summary

We have introduced three mathematical tools for exploring scalar quantities and vector fields.

- We have the **gradient operator**, $\boldsymbol{\nabla}(\,)$ for use in determining how a quantity, either a vector component or scalar, varies in an elevation-like way at any place. This is like finding the slope of a function.
- We have the **divergence operator**, $\boldsymbol{\nabla} \cdot \vec{(\,)}$, for use in determining whether at a place within a vector field confined by an infinitesmal volume, it is either increasing or decreasing in magnitude.
- We have the **curl operator**, $\boldsymbol{\nabla} \times \vec{(\,)}$, for use in determining whether and by how much and according to which orientation a vector field twists or swirls.

I would like you to know when you might want to use each mathematical tool in your explorations of scalar variations or of a vector field. Still, as we generally reduce our problems and mathematical developments in lower level courses to 1-dimension, the details of these operators may not have appeared as useful in your previous courses. They were of some importance, though, in *Earthquakes and Earth Structures.*

4 The prototypical equations of geophysics

Classical physics and geophysics are largely contained by the understanding of only a few *partial differential equations.* These equations, typically, relate spatial variations of some (geo)physical quantity to possible temporal (i.e. time) variations of the quantity.

Before we look to these, though, and without elaborating the details, here, it is perhaps interesting to note that we might obtain the *divergence* of that vector quantity that we obtained as the *gradient* of some scalar or vector component. One obtains the **Laplacian** of a scalar function or vector component function as:

$$\boldsymbol{\nabla}^2(\,) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}(\,).$$

We might obtain the *gradient* of the *divergence* of some vector quantity. Again, though with some restrictions:

$$\boldsymbol{\nabla^2}(\vec{\,}) = \boldsymbol{\nabla\nabla\cdot}(\vec{\,}).$$

We might also obtain the *curl* of the *curl* of some vector field. Again, surprisingly:

$$\boldsymbol{\nabla^2}(\vec{\,}) = \boldsymbol{\nabla\times\nabla\times}(\vec{\,}).$$

What of the *divergence* of the *curl* of a vector field, the *curl* of the *divergence* of a vector field and the *curl* of the *gradient* of some scalar field or field of vector components? Usually (though special conditions must hold for some of these):

$$\boldsymbol{\nabla\cdot\nabla\times}(\vec{\,}) = \boldsymbol{\nabla\times\nabla\cdot}(\vec{\,}) = \boldsymbol{\nabla\times\nabla}(\,) \equiv \mathbf{0}.$$

4.1 Laplace's equation and Poison's equation

Many geophysical fields, such as the gravitational potential in regions where there is no mass contributing to the gravitational potential, obey **Laplace's equation**:

$$\boldsymbol{\nabla^2}(\boldsymbol{U}) = \mathbf{0}.$$

This brings a powerful convenience to geophysical mapping. For example, if we were to map the gravitational potential (It works as well for the vertical acceleration of gravity.) over a surface region of the Earth, we could determine how the gravitation potential varied with elevation from that surface. We could *downward continue* or *upward continue* the field. We shall deal with these techniques later in the course; they are especially useful in *exploration geophysics* which is the topic of our course *Geophysical Applications*.

In regions where there is mass contributing to the gravitational potential, as within the body of the Earth, **Poisson's equation** holds as:

$$\boldsymbol{\nabla^2}(\boldsymbol{U}) = \mathbf{4}\boldsymbol{\pi G\rho}.$$

where in this case $\boldsymbol{G} = 6.67259(30) \times 10^{-11}\ m^3 \cdot kg^{-1}s^{-2}$ is the current IAU *International Astronomical Union's* value for the *Cavendish gravitational constant* of Newtonian and relativistic mechanics. $\boldsymbol{\rho}$ $[kg \cdot m^{-3}]$ is the local mass density in the Earth.

Laplace's equation, $\boldsymbol{\nabla^2}(\,) = \mathbf{0}$, can reasonably be considered as a special case of Poisson's equation wherein the constant is **0**.

4.2 The diffusion equation

A potential field might vary with time. If we had some potential function, like the concentration, $\boldsymbol{C}$, of a particular element, we might recognize that over time it might diffuse throughout a medium into regions of lower concentration. The rate of this *diffusion* is related to the laplacian of the spatial distribution of the potential as:

$$\boldsymbol{\nabla^2(C) = D\partial_t(C),}$$

where $\boldsymbol{D}$ represents the *diffusivity* related to the rate of diffusion.

Note that our physical units must be the same for all *terms* in an equation. The left side of this equation is twice differentiated with respect to the spatial variables and there is a single differentiation with respect to time on the right side. If we were to measure $\boldsymbol{C}$ in, for example, $moles \cdot litre^{-1}$, the units appropriate to the left side become $moles \cdot litre^{-1} \cdot metre^{-2}$ and noting that $1litre = 10^{-3}metres^3$, we can resolve the left-side units as $moles \cdot m^{-5}$. The units of the constant can easily, then, be found $\boldsymbol{D}{-}units \times moles \cdot m^{-3}s^{-1} = moles \cdot m^{-5}$ through simple algebra on the units themselves. The units of the constant of diffusivity, $\boldsymbol{D}$, are then $m^{-2} \cdot s$.

4.3 The wave equation

The field variable, let's say $\boldsymbol{\Phi}$, representing the seismic P-wave potential function varies spatially as the *laplacian* but temporally according to a twice-differentiation with respect to time:

$$\boldsymbol{\nabla^2(\Phi) = \frac{1}{\alpha^2}\partial_t^2(\Phi),}$$

where

$$\boldsymbol{\alpha = \sqrt{\frac{k + \frac{4}{3}\mu}{\rho}}}$$

is the speed of the seismic P-wave with units $[m \cdot s^{-1}]$. Here $\boldsymbol{k}$ is the *bulk incompressibility*, $\boldsymbol{\mu}$, the *rigidity*, both with units $[Pa]$ and $\boldsymbol{\rho}$, units $[kg \cdot m^{-3}]$, the *density*, locally, in the material through which the wave is travelling.

4.4 The diffusive wave equation

Perhaps it is not surprising that we might have potentials that travel as waves and also which are diffusive:

$$\boldsymbol{\nabla^2(\Phi) = \frac{1}{\alpha^2}\partial_t^2(\Phi) + D\partial_t\Phi.}$$

This form is sometimes (poorly, though) used to model waves that decay with time and travel through a material. We leave details to the *Earthquakes and Earth Structures* course.

Note that the latter three equations (properly all 5 equations) are **"homogeneous"** *partial differential equations* which means that they do not contain any real constant terms. Note, as well, that these equations are all **linear** in that no term is quadratic (squared) or greater. We shall not, here, infinitely complicate issues with non-linearities and inhomogeneities. We touch on these issues, of course, in other courses: *Tectonics*, *Earthquakes and Earth Structures* and possibly in *Geophysical Applications*.

4.5 Potential fields

In physics, we have learned to construct mathematical forms called *potential functions* which, upon some operation, describe measurables. Properly, potentials are fiction-like in that we can't measure them directly. We can, though, often measure some derived property. For example, we can construct a gravitation potential due to a point mass, $\boldsymbol{M}$, as

$$\boldsymbol{U(\vec{r}) = -\frac{GM}{|\vec{r}|}}.$$

This potential function varies with place, $\vec{\boldsymbol{r}}$, but is purely scalar. We can't directly measure the potential but we can measure its *gradient*:

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}) = -\boldsymbol{\nabla U}(\vec{\boldsymbol{r}})$$

is the gravitational accleration per unit mass due to that gravitational potential at position $\vec{\boldsymbol{r}}$. Why potential field theory is so powerful and useful is that potentials are simply additive. The potential function of two point masses is simply the sum of the individual potentials of each mass.

In the equations above, $\boldsymbol{\Phi}(\vec{\boldsymbol{r}})$ in the wave equation and diffusive wave equation is the *P-wave potential*. The *divergence* of that potential is measurable as the immediate and local change in the *cubical dilatation*, $\boldsymbol{\Delta V/V}$, caused by the wave. If this P-wave is a sound wave in air, for example, this cubical dilatation is related to a change in pressure – that which affects our ear's tympanum – as $\boldsymbol{\Delta P = k\Delta V/V}$ where $\boldsymbol{k}$ is a measure of the *bulk incompressibility* of air.

We can construct both vector and scalar potentials for magnetics as well as the scalar potential for gravity. We form a *magnetic potential*, $\vec{\boldsymbol{A}}$, for temporally varying magnetic phenomena such that the measurable *magnetic induction* $\vec{\boldsymbol{B}} = \boldsymbol{\nabla} \times \vec{\boldsymbol{A}}$. We can also form a *magnetostatic potential*, $\boldsymbol{V}$, for non-temporally varying magnetic phenomena such that $\vec{\boldsymbol{B}} = \boldsymbol{\nabla V}$.

4.6 Schröedinger's wave equation

In the 1920s, physicists developed the theoretical basis of Quantum Mechanics. It is essentially based in an equation defining the ***wave function*** $\boldsymbol{\Psi(r, t)}$. For a single particle of mass $\boldsymbol{m}$ moving about in 3-dimensional space constrained by local potential $\boldsymbol{V}$, its wave function is described by Schröedinger's equation:

$$\boldsymbol{\nabla^2 \Psi + \frac{4\pi i m}{h} \frac{\partial \Psi}{\partial t} + \frac{8\pi^2 m}{h^2} V \Psi = 0.}$$