Geomagnetics

1 Basic EM concepts

Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, Ohm's law and the constitutive relationships form the foundation of classical electrodynamics, classical optics, and electric circuits. These fields in turn underlie modern geo-electromagnetics.

1.1 The vector fields

Electromagnetic phenomena are best and conventionally described by a suite of vector fields and their inter-relationships.

- **The Electric field $\vec{E}$:** The electric field describes, in the 3-space, the equivalent of the electrical potential (voltage) in the one-space world of electric circuits. Generalized to the 3-space, the field has direction and magnitude. In SI units, the magnitude is measured in $[V \cdot m^{-1}]$.
- **The Magnetic field $\vec{H}$:** The magnetic field (in some descriptions called the "magnetizing field") describes an analog of the current in the one-space world of electrical circuits. You might note that the magnetic field describes, essentially, a current per unit distance in the 3-space; in SI units, the magnitude of the magnetic field vector is measured in $[A \cdot m^{-1}]$.
- **The Magnetic induction $\vec{B}$:** The magnetic induction represents the effect of the magnetic field in a material or space having magnetic permeability. In SI physics, it is measured in units of $[V \cdot s \cdot m^{-2}]$ or $[T]$, "tesla". The magnetic permeability of "free space" (an idealized perfect vacuum) is and invariant in contemporary SI-physics, fixed to be $\boldsymbol{\mu_0 = 4\pi \times 10^{-7}}$ $[V \cdot s \cdot A^{-1} \cdot m^{-1}]$. One might regard this field as measuring the local "magnetic polarization". In free space $\vec{B} = \mu_0 \vec{H}$.
- **The Electric displacement $\vec{D}$:** The electric displacement (sometimes called dielectric displacement) is the electric field's analog of $\vec{B}$. In SI physics, it is measured in units of $[kg \cdot s^{-2} \cdot V^{-1}]$. It measures, again, the local "electric polarization" according to the "dielectric permittivity" which is a derived or measured quantity in SI physics: $\vec{D} = \epsilon_0 \vec{E}$ in free space. The best current value for $\boldsymbol{\epsilon_0 = 8.854187817620... \times 10^{-12}}$ $[A^2 \cdot s^4 \cdot kg^{-1} \cdot m^{-3}]$, determined as

$$\boldsymbol{c_o^2 = \frac{1}{\mu_0 \epsilon_o},}$$

where $\boldsymbol{c_o}$ is the measured "speed of light" in free space.

1.2 Relationships among these vector fields: Maxwell's Equations

In one of the most beautiful of all physical descriptions, Maxwell showed how these vector fields are related one to another.

- **Gauss' law for electrics:**
$$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{D}} = \boldsymbol{\rho_f}.$$
Here, $\boldsymbol{\rho_f}$ represents the local density of "free charges" measured in units of $[C \cdot m^{-3}]$ where $[C]$ is the "coulomb", the common SI-unit of charge.

- **Gauss' law for magnetics:**
$$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{B}} = \mathbf{0}.$$
You might note the similarity with Gauss' law for electrics; electrical "charges" are separable in $+$ and $-$ charges while magnetic "poles" have never been found to be separable in nature, always found locally in balanced pairings of $\boldsymbol{N}$ and $\boldsymbol{S}$ poles of equal strength.

- **Faraday's law of induction:**
$$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = \frac{\boldsymbol{\partial} \vec{\boldsymbol{B}}}{\boldsymbol{\partial t}},$$
showing how an electrical field can be generated through temporal variations in the magnetic induction.

- **Ampère's law:**
$$\boldsymbol{\nabla} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}}_{\boldsymbol{f}} + \frac{\boldsymbol{\partial} \vec{\boldsymbol{D}}}{\boldsymbol{\partial t}},$$
showing how a magnetic field is generated by a current flow and temporal variations in the electrical displacement. In highly conductive materials like metals,
$$\frac{\boldsymbol{\partial} \vec{\boldsymbol{D}}}{\boldsymbol{\partial t}} \longrightarrow \mathbf{0}$$
extremely rapidly. Above, $\vec{\boldsymbol{J}}_{\boldsymbol{f}}$ is the local "free current density" measured in units of $[A \cdot m^{-2}]$.

- **Ohm's law**:

$$\vec{J_f} = \sigma \vec{E}$$

  where σ describes the local "conductivity" measured in $[S \cdot m^{-1}]$ where the "seimen" is a unit of conductivity related to the common unit of resistivity, often designated ρ, the "ohm", as $1S = 1\Omega^{-1}$. Note that this ρ is not the ρ_f that represented the local free charge above!

1.2.1 Relationships with currents and fields:

For some phenomena in geoelectromagnetics, we can ignore the direct magnetic fields that are always involved with electrical current flows. By convention, current flows from a high electrical potential to a lower potential. In terms of fields, current flows locally through isotropic conductors along the $\vec{E}$ field direction. Ohm's law in its simplest form as described above for a material of uniform and isotropic conductivity, σ, must be generalized to account for local anistropy:

$$J_{f_j} = \sigma_{ij} E_j$$

where, now, the anisotropic conductivity is described as a 2-tensor. The current flow direction is, now, not necessarily parallel to the $\vec{E}-$field.

An aside of some importance in geophysical electromagnetics: Usually in courses in EM-theory, one is led to look at the various fields as being either constant for all frequencies of variation or, alternatively, as being of a single frequency. For example, we might more completely write Ohm's law above as:

$$J_{f_j}(\omega) = \sigma_{ij}(\omega) E_j(\omega)$$

where ω represents a particular frequency of observation. The rule can hold frequency-by-frequency allowing for a frequency dependence in conductivity. This is most commonly the case in geoelectromagnetics.

One might also look to Ohm's law in the time domain where, now,

$$J_{f_j}(t) = fnc\big(\sigma_{ij}(t), E_j(t)\big)$$

which just might be, exceptionally, a linear functional relationship as, for example,

$$J_{f_j}(t) = \sigma_{ij}(t) E_j(t)$$

allowing for the time and frequency domain descriptions to have, pretty much, the same character. There is a subtlety though that must be addressed when looking to natural phenomena in the time domain of measurement and in the frequency domain of easy description: the functional relationship, even in the simple linear case, must be "causal". That is, until the $\vec{E}-$field appears, there must be no resultant $\vec{J_f}-$field described. This leads to the requirement that the real and imaginary parts of the complex-valued conductivity tensor (already complicated) form a "Hilbert transform" pair. In some courses in EM-theory, you might have been briefly exposed to this requirement as the Kramers-Kroenig relationship for EM-fields. In geophysical principle, or observation, a further requirement also is seen to hold. Fermat's principle, possibly only second in importance to the principle of causality[1] in physics. Fermat's principle, in this story seen as a "minimum-phase" requirement, determines further that the logarithms of the real and imaginary parts of the conductivity tensor obey Hilbert-transform pairing. This story is well beyond the scope of this course.

For interest, you might delve into Hilbert transform theory. Similar requirements hold for material dielectric permittivity, $\boldsymbol{\epsilon}$, and magnetic permeability, $\boldsymbol{\mu}$. Such details are important in applied geophysical techniques like IP (induced polarization), MT (magneto-tellurics), MR (magnetic relaxation) methods[2].

1.2.2 Material magnetics and dielectrics:

In analogy to Ohm's law, just elaborated, magnetic and dielectric material properties require some further description. These properties relate the $\vec{B}$ and $\vec{H}-$fields through a material's magnetic permeability and the $\vec{D}$ and $\vec{E}$ fields through its dielectric permittivity:

$$\vec{B} = \mu\vec{H}, \qquad \vec{D} = \epsilon\vec{E}.$$

Again, very simple models of $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ are commonly described in physics courses in EM-theory whereas in geophysical reality all the complexities described for conductivity above arise as well. Usually,

$$\epsilon = \kappa_\epsilon \epsilon_0$$

and

$$\mu = \kappa_\mu \mu_0$$

[1]Note that "causality" does not imply "agency" in physics; that is, it is not necessary that some agent causes a response. This is the inherent base of Quantum Mechanics and contemporary cosmologies. The Big-Bang explosion of the universe needs no "god-agent" to push the button to make it happen. It was, by most cosmological models, a purely "spontaneous" event. I suggest that you can guess whether or not a physicist is a theist by asking the question: "*What caused the universe?*" though properly, such questions are the realm of meta-physical philosophy.

[2] Baranyi, Jensen and LaFleche, 1987, Inverse solution modeling of the complex-valued, frequency dependent electrical properties of natural geologic materials, RADIO SCIENCE, VOL. 22, NO. 4, PP. 511-519

in a description of material properties as being naught but a scaling of those of the vacuum free space. In this simple scaling model,

$$\kappa_\epsilon = (1 + \chi_\epsilon)$$

where χ_ϵ measures the internal dielectric polarization component due to the $\vec{E}-$field and

$$\kappa_\mu = (1 + \chi_\mu)$$

where χ_μ measures the internal magnetic polarization component due to the $\vec{H}-$field. χ_ϵ and χ_μ may be either positive or negative quantities, usually small. For example "paramagnetic" and "paraelectric" materials show positive χs while "diamagnetic" and "diaelectric" materials show negative χs. Again, if passionately interested, you might see the paper by Elizabeth Baranyi *et al.* listed previously.

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More composition of the the story to come...

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1.3 How a dynamo works!

The Lorentz forcing of a current arises when a conductor is moved through a magnetic field. The force on a charge q moving with velocity $\vec{v}$ through a field of magnetic induction, $\vec{B}$, and an electric field, $\vec{E}$,is obtained as

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}].$$

We might regard a current as a forced flow of positive charges even though we might know that it is a reverse drift of negative free electrons. In a Faraday homopolar dynamo, the forcing should cause a current flow toward the outer rim of the rotating conductive disk. If we can extract this current into an external circuit coil aligned so as to "augment" the field of magnetic induction, we could produce a dynamo that is self-sustaining providing the power required to maintain the rotation is offered[3].

[3] Faraday's paradox

Homopolar dynamo description