Spherical harmonic expansions of geo-fields

1 Laplace's equation

Earlier in the term, we encountered the classical equations of geophysical fields and their dynamics. That which is simplest among them is Laplace's equation. This equation holds for static fields in what is sometimes called the "harmonic domain". The equation for a harmonic field – let's call it V – is:

$$\nabla^2 V = 0.$$

Earlier in the year, we described non-harmonic and dynamic variations on this equation, for example, Poisson's equation,

$$\nabla^2 V = C,$$

the wave equation,

$$\nabla^2 V = \frac{\partial^2 V}{\partial t^2},$$

and the diffusion equation,

$$\nabla^2 V = \frac{\partial^2 V}{\partial t^2}.$$

Wherever Laplace's equation holds, if we know V over a surface in that space, we can obtain V on any other surface in that space. For example, if we were to describe $V(x, y, z)$ on a surface $z = z_1$, we can determine $V(x, y, z = z_2)$ through simple harmonic continuation. Suppose know the surface $V(z = 0)$ where 0 is a chosen arbitrary datum level in z. Form:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\partial^2 V}{\partial z^2}$$

and obtain the Fourier transform of this equation with respect to x and y:

$$-[k_x^2 + k_y^2]\mathcal{V}(k_x, k_y, z) = -\frac{d^2}{dz^2}\mathcal{V}(k_x, k_y, z).$$

We now have an ordinary, second order differential equation in $\mathcal{V}$ which has general solutions of the form

$$\mathcal{V}(k_x, k_y, z) = \mathcal{A}(k_x, k_y)e^{\sqrt{k_x^2+k_y^2}z} + \mathcal{B}(k_x, k_y)e^{-\sqrt{k_x^2+k_y^2}z}.$$

Expecting that our gravitational potential $V(x, y, z)$ is due to some local "source", as we increase our distance from that source, V increases toward 0 by convention. Normally we describe a reference potential field with value $V = 0$ at infinite distance from the field source. That means that $\mathcal{V}(k_x, k_y, z = \infty) \to 0$

and, consequently, $\mathcal{A}(k_x, k_y) = 0$. It then becomes trivial to change the "elevation" of observation... Fourier transform the potential function $V(x, y, z_1)$ to obtain $\mathcal{B}(k_x, k_y)e^{-\sqrt{k_x^2+k_y^2}z_1}$, then, for surface at $z = z_2$, reform $\mathcal{V}(k_x, k_y, z_2)$ with the amplitude coefficients $\mathcal{B}(k_x, k_y)$ and invert the Fourier transform for $V(x, y, z_2)$.

In geophysical exploration, upward and downward continuation are seldom applied to the potentials themselves. Rather, we recognize that the gradients of the potentials and the components of the gradients also follow Laplace's equation such that if the anomalous gravity acceleration field is $\vec{g}_a(x, y, z) = -\nabla V_a(x, y, z)$ and if we upward continue the vertical component of gravity, g_{a_z} from a datum surface, say $z = 0$ to another surface at elevation $z > 0$, we simply multiply Fourier coefficients, $\mathcal{G}_{a_z}(k_x, k_y, z = 0)$ by $e^{-\sqrt{k_x^2+k_y^2}z}$ and invert the scaled Fourier transform. For downward continuation from surface $z > 0$ to $z = 0$, we would simply divide by the same scale coefficients.

These field continuation rules hold wherever Laplace's equation holds for the field being continued and in whatever coordinate system we choose to "measure" its values. The process is not essentially more difficult to realize for a spherical coordinate system though the mathematical technology appears more complicated. We can upward or downward continue the gravitational potential field of the Earth or the static geomagnetic potential field as described by spherical harmonic coefficients providing we do not attempt to "continue through sources". That is for gravity, anywhere above the surface of the Earth, Laplace's equation holds at least approximately as we ignore the atmospheric density variations. For the geomagnetic field, the surface field can be upward continued through to the base of the ionosphere where electrojet currents contribute to sources. Normally, in continuing the geomagnetic field, we attempt to "model out" the ionospheric current field. This can be a difficult problem in situations where, for example, strong auroral displays are being observed.

2 Expansion for the "geoid"

The stationary part of the Earth's gravitational potential U at any point $P(r, \phi, \lambda)$ on and above the Earth's surface is expressed on a global scale conveniently by summing up over degree l and order $m = 0$ of a spherical harmonic expansion. The spherical harmonic (or Stokes') coefficients represent in the spectral domain the global structure and irregularities of the geopotential field or, more generally spoken, of the gravity field of the Earth. The equation relating the spatial and spectral domain of the geopotential is as follows:

$$U(r, \phi, \lambda) = \frac{GM}{R}\left[\frac{R}{r}C_{00} + \sum_{l=1}^{l_{max}}\sum_{m=0}^{l}\left(\frac{R}{r}\right)^{l+1} P_{lm}(\sin\phi)(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda)\right]$$

where

r, ϕ, λ are the spherical geocentric coordinates of computation point (radius, latitude, longitude),
R is the reference length (mean semi-major axis of Earth),
GM is the gravitational constant times mass of Earth, l, m, degree, order of spherical harmonic,
P_{lm} are the fully normalized Lengendre functions, and
C_{lm} and S_{lm} are the Stokes' coefficients (fully normalized).

C_{00} is close to 1 and scales the value GM. The degree 1 spherical harmonic coefficients $C_{1\times}$ are related to the geocentre coordinates and zero if the coordinate systems' origin coincides with the geocentre. The coefficients, C_{21} and S_{21} are connected to the mean rotational pole position that is a function of time.

The Earth's shape is approximately ellipsoidal with Stoke's coefficients $C_{n0}^{ell}, n = 0, 2, 4, \ldots$. The expansion with these coefficients that most closely fits the surface of the Earth is called the reference potential, $V(r, \phi, \lambda)$. The difference between $V(r, \phi, \lambda)$ and $U(r, \phi, \lambda)$ for $r = R$, the surface, determines the geoidal deviations from the reference ellipsoid:

$$T(R, \phi, \lambda) = \frac{GM}{R}[C'_{00} + \sum_{l=1}^{l_{max}}\sum_{m=0}^{l} P_{lm}(\sin\phi)(C'_{lm}\cos m\lambda + S'_{lm}\sin m\lambda)]$$

with $C' = C - C^{ell}$. You might note that C'_{00} is very nearly 0. This is essentially the spherical harmonic expansion of the "geoid". $T(r, \phi, \lambda)$ is the "disturbing potential" field that a satellite at radial distance r would feel that differs from that of a uniform ellipsoidal Earth. We may map this at elevation of the satellite and then downward continue it to the surface where $r = R$. Downward continuation is "easy". One need only scale each coefficient by $(\frac{r}{R})^{l+1}$.

3 Expansion for the geomagnetic field

A major difference between the gravitational expansion and that for magnetism derives from the fact that magnetic poles must always come in pairs. That is there is no monopolar field component. If $W(r, \phi, \lambda)$ represents the full harmonic expansion of the geomagnetic field, that expansion can have no C_{00}, monopolar, component. Moreover, all coefficients for odd l should also be zero.

$$W(r, \phi, \lambda) = Q\sum_{l=1}^{l_{max}}\sum_{m=0}^{l}(\frac{R}{r})^{l+1} P_{lm}(\sin\phi)(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda)$$

where $\boldsymbol{Q}$ represents the scale of the field strength. The Earth's "dipole moment", commonly used in the scaling of a planetary magnetic field, is determined by $\boldsymbol{\nabla W(r, \phi, \lambda); l_{max} = 1}$. Given this expansion, you might note that the dipole potential field falls off with distance as $\boldsymbol{r^{-2}}$; the gradient of this field, the measurable quantity, falls of as $\boldsymbol{r^{-3}}$. The monopolar field of gravitational potential falls off as $\boldsymbol{r^{-1}}$, it gradient, the measurable acceleration falls off as $\boldsymbol{r^{-2}}$.