Geophysical electromagnetics and magneto-tellurics

1 Theory: Maxwell's equations

In 1861-62, James Clerk Maxwell formalized the results of a series of experiments in electricity and magnetism into one coherent and elegant mathematical theory of electromagnetism.

In our current notational style, Maxwell's theory is described in terms of four partial differential equations relating electric and magnetic fields and free charges and possible magnetic monopoles within space and materials.

- **Gauss' Law:**
$$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{D}} = \boldsymbol{\rho_e}$$

- **Gauss' Law for magnetism:**
$$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{B}} = \mathbf{0},$$
assuming that free magnetic monopoles do not exist. If free monopoles do exist, their presence modifies this law:
$$\boldsymbol{\nabla} \cdot \vec{\boldsymbol{B}} = \boldsymbol{\rho_m}.$$
Free magnetic monopoles have never been detected, even at the highest energies of experiments in particle accelerators. Perhaps the *Large Hadron Collider* will detect them.

- **Faraday's Law (induction):**
$$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial \boldsymbol{t}}.$$
Again if free monopoles do exist, one might expect a monopole current density, say $\vec{\boldsymbol{K}}$ that would add to the equation describing the law as
$$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} = \vec{\boldsymbol{K}} - \frac{\partial \vec{\boldsymbol{B}}}{\partial \boldsymbol{t}}.$$

- **Ampere's Law:**
$$\boldsymbol{\nabla} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{D}}}{\partial \boldsymbol{t}}.$$

In these four equations, $\vec{\boldsymbol{D}}$ is the *dielectric displacement vector*, $\vec{\boldsymbol{B}}$ is the *magnetic induction vector*, $\vec{\boldsymbol{E}}$, the *electric field vector*, $\vec{\boldsymbol{H}}$, the *magnetic field vector*, $\vec{\boldsymbol{J}}$, the *current density* of electric charges and, if monopoles do exist, $\vec{\boldsymbol{K}}$, the *current density* of possible magnetic poles.

- **The constitutive relationships:** The volume in which these fields exist relates the vector fields to each other through the constitutive equations.

$$\vec{\boldsymbol{D}} = \boldsymbol{\epsilon}\vec{\boldsymbol{E}},$$

$$\vec{\boldsymbol{B}} = \boldsymbol{\mu}\vec{\boldsymbol{H}},$$

  and if the space in which these fields exist is the *free space* perfect vacuum, $\boldsymbol{\epsilon} \longrightarrow \boldsymbol{\epsilon_0}$ and $\boldsymbol{\mu} \longrightarrow \boldsymbol{\mu_0}$. If the space is filled with materials that are electromagnetically isotropic and that respond linearly to variations in the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ fields, $\boldsymbol{\epsilon} = \boldsymbol{\kappa_\epsilon \epsilon_0}$ and $\boldsymbol{\mu} = \boldsymbol{\kappa_m \mu_0}$. If the material properties are not isotropic, then $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ (or, $\boldsymbol{\kappa_\epsilon}$ and $\boldsymbol{\kappa_\mu}$) are described as 2-tensors rather than scalars. If the materials do not respond linearly, then we could require very much more complex relationships between the four fields.

To the above 6 relationships, we often add, yet another:

- **Ohm's law:**

$$\vec{\boldsymbol{J}} = \boldsymbol{\sigma}\vec{\boldsymbol{E}}$$

  where $\boldsymbol{\sigma}$ is the material conductivity which might be a scalar or 2-tensor quantity and may not be describable by a simple linear relationship between the fields.

In the following development, we shall retreat to the simplicity of isotropy and linearity of the constitutive relationships and Ohm's law.

1.1 Boundary conditions

Generally, the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ vectors that describe an electromagnetic field or wave at any point must be continuous functions of space and time. Across a boundary between two materials with different physical properties according to $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$, the field vectors, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$, remain continuous. Usually, the boundary condition is sufficiently established if the components of $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ locally tangential to the boundary are continuous. While solutions are often mathematically complex, knowing the geometry and scale of a structure and the variations in $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ across boundaries and within the structure, we can, in principle, describe the relationships between the field vectors.

We often distinguish *near-field* and *far-field* descriptions of electromagnetic wave phenomena. The near-field description deals with the region close to the source field and is generally more complicated to describe than the far-field region where a simpler wave description holds. In the far-field we can describe an electromagnetic wave with a wave theory not unlike that we have described for seismic S-waves. The wave is modelled as oscillating in time (with angular frequency $\boldsymbol{\omega}$) and in space (with propagation vector $\vec{\boldsymbol{k}}$). Locally, the wave may be described as:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},\boldsymbol{t}) = \vec{\boldsymbol{\mathcal{E}}}(\vec{\boldsymbol{k}},\boldsymbol{\omega})\boldsymbol{e}^{\boldsymbol{i}(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\boldsymbol{\omega t})}$$

and

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{r}},\boldsymbol{t}) = \vec{\boldsymbol{\mathcal{H}}}(\vec{\boldsymbol{k}},\boldsymbol{\omega})\boldsymbol{e}^{\boldsymbol{i}(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\boldsymbol{\omega t})}$$

where the propagation vector may, itself, vary in time and space: $\vec{\boldsymbol{k}}(\vec{\boldsymbol{r}},\boldsymbol{t})$**.** Generally, the wave amplitudes, $\vec{\boldsymbol{\mathcal{E}}}(\vec{\boldsymbol{k}},\boldsymbol{\omega})$ and $\vec{\boldsymbol{\mathcal{H}}}(\vec{\boldsymbol{k}},\boldsymbol{\omega})$, are complex valued and orthogonal to one another. $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{H}}$ locally relate to one another according to Maxwell's equations. Being complex valued, we often describe the amplitude vectors in a polar form as:

$$\vec{\boldsymbol{\mathcal{E}}}(\vec{\boldsymbol{k}},\boldsymbol{\omega}) = |\vec{\boldsymbol{\mathcal{E}}}|\boldsymbol{e}^{\boldsymbol{i\phi_E}}$$

and

$$\vec{\boldsymbol{\mathcal{H}}}(\vec{\boldsymbol{k}},\boldsymbol{\omega}) = |\vec{\boldsymbol{\mathcal{H}}}|\boldsymbol{e}^{\boldsymbol{i\phi_H}}$$

where the $\boldsymbol{\phi}$s are the phase angles referenced to some time origin. Of course,

$$\boldsymbol{\phi_E} = \mathbf{tan}^{-1}\,(\frac{\Im(\vec{\boldsymbol{\mathcal{E}}})}{\Re(\vec{\boldsymbol{\mathcal{E}}})}),$$

$$\boldsymbol{\phi_H} = \mathbf{tan}^{-1}\,(\frac{\Im(\vec{\boldsymbol{\mathcal{H}}})}{\Re(\vec{\boldsymbol{\mathcal{H}}})}).$$

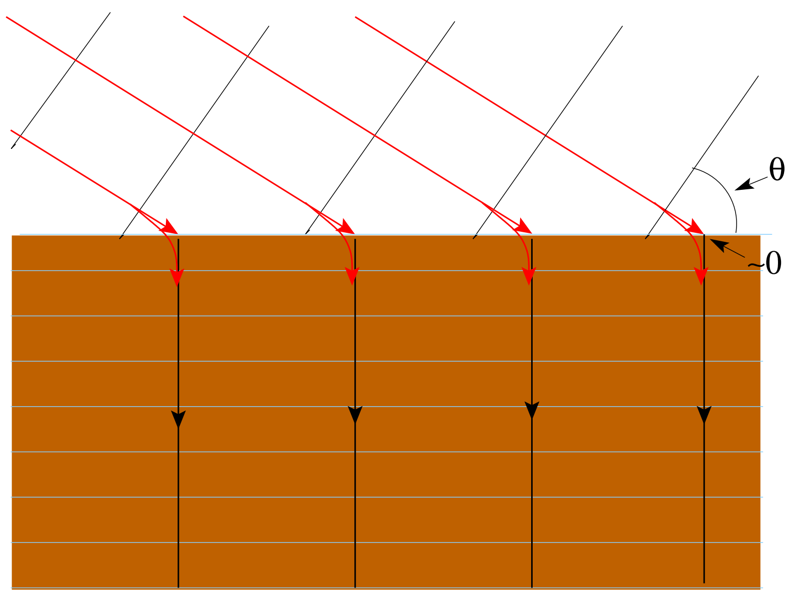
The symbols $\Re$ and $\Im$ determine the *real* and *imaginary* parts of the complex value of argument.

2 Simple magneto-tellurics

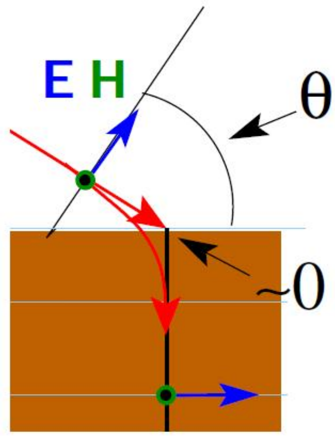
Near field (i.e. close to the source of the EM field) technologies are important in geophysical exploration. Most geophysical surveying methods involving electromagnetics involve near-field effects of conductivity, permittivity and permeability of geological materials in response to imposed, temporally varying electromagnetic fields. A detailed description of these surveying technologies is left to our course in *Geophysical Applications – EPSC435.* Magneto-telluric methods are also useful in geophyscal surveying though our reason for including them here is that there exist a wide spectrum of natural magneto-telluric fields and waves that should, properly, be addressed in a course in Earth Physics. The boundary between *Applied Geophysics* and *Earth Physics* is diffuse here.

Natural, time varying electromagnetic fields and waves are produced by currents in the Earth's ionosphere (the plasma region of the upper atmosphere) and by lightning strikes on Earth that radiate electromagnetic waves into the spherical-shell waveguide between the base of the ionosphere and the surface of the Earth. These fields and waves are significantly influenced by the physical properties, ($\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$), of the solid Earth. We address the simplest of these magneto-telluric problems.

Consider a flat-Earth halfspace characterized by $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ into which an electromagnetic wave enters from the halfspace above. While that upper halfspace properly models the near-surface troposphere of the Earth, let us replace it with a free-space: $\boldsymbol{\epsilon_0}$, $\boldsymbol{\mu_0}$ and $\boldsymbol{\sigma_0 = 0}$.



Note that electromagnetic waves are transverse vector waves; the field vectors, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, \boldsymbol{t})$ and $\vec{\boldsymbol{H}}(\vec{\boldsymbol{r}}, \boldsymbol{t})$, are, therefore, parallel to the plane of the wavefront and orthogonal to one another. As long as these conditions are satisfied, they may be polarized in any compatible way. One simple polarization might be with the $\vec{\boldsymbol{H}}(\vec{\boldsymbol{r}}, \boldsymbol{t})$ horizontally aligned on the wavefront, oscillating in and out of the diagram. This would describe a linear, horizontal $\vec{\boldsymbol{H}}$-polarization. The $\vec{\boldsymbol{E}}$ would, then, lie in the plane of the diagram along a wavefront.



2.1 Electromagnetic waves

Recalling the following of Maxwell's equations,

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{E}} + \frac{\partial \vec{\boldsymbol{B}}}{\partial t} = 0,$$

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{H}} - \frac{\partial \vec{\boldsymbol{D}}}{\partial t} = \vec{\boldsymbol{J}},$$

we obtain the curl ($\boldsymbol{\nabla}\times$) of both as

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\boldsymbol{E}} + \frac{\partial(\boldsymbol{\nabla} \times \vec{\boldsymbol{B}})}{\partial t} = 0,$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\boldsymbol{H}} - \frac{\partial(\boldsymbol{\nabla} \times \vec{\boldsymbol{D}})}{\partial t} = \boldsymbol{\nabla} \times \vec{\boldsymbol{J}}.$$

Noting that $\vec{\boldsymbol{B}} = \boldsymbol{\mu}\vec{\boldsymbol{H}}$, $\vec{\boldsymbol{D}} = \boldsymbol{\epsilon}\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{J}} = \boldsymbol{\sigma}\vec{\boldsymbol{E}}$, and if in our space of interest $\boldsymbol{\mu}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ do not vary with time or place, we rewrite,

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\boldsymbol{E}} + \mu \frac{\partial(\boldsymbol{\nabla} \times \vec{\boldsymbol{H}})}{\partial t} = 0,$$

substitute for $\boldsymbol{\nabla} \times \vec{\boldsymbol{H}} = \boldsymbol{\epsilon}\partial\vec{\boldsymbol{E}}/\partial t - \boldsymbol{\sigma}\vec{\boldsymbol{E}}$ to obtain

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\boldsymbol{E}} + \mu\epsilon \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} + \mu\sigma \frac{\partial \vec{\boldsymbol{E}}}{\partial t} = 0.$$

Since $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\boldsymbol{A}} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \vec{\boldsymbol{A}} - \boldsymbol{\nabla}^2 \vec{\boldsymbol{A}}$ and if there is no free charge density as there cannot be in any conductive material, $\boldsymbol{\nabla} \cdot \vec{\boldsymbol{D}} = \boldsymbol{\epsilon}\boldsymbol{\nabla} \cdot \vec{\boldsymbol{E}} = 0$,

$$\boldsymbol{\nabla}^2 \vec{\boldsymbol{E}} - \mu\epsilon \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} - \mu\sigma \frac{\partial \vec{\boldsymbol{E}}}{\partial t} = 0.$$

Through similar arguments, noting that, generally, $\nabla \cdot \vec{B} = \mu \nabla \cdot \vec{H} = 0$, we also obtain

$$\nabla^2 \vec{H} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \frac{\partial \vec{H}}{\partial t} = 0.$$

We recognize these as two diffusive wave equations.

2.1.1 $\sigma = 0$

In a *"non-conductive medium"*, $\sigma = 0$ and these equations reduce as:

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0,$$

$$\nabla^2 \vec{H} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0.$$

You might recognize these as describing a non-diffusive wave characterized by a speed of propagation,

$$c = \frac{1}{\sqrt{\mu\epsilon}}$$

and in free space

$$c_o = \frac{1}{\sqrt{\mu_o \epsilon_o}}.$$

You might recall from our earliest lectures that $\mu_o = 4\pi \times 10^{-7}\ Wb \cdot A^{-1} \cdot m^{-1}$. c_o is now a defined constant in physics which determines, among other things, the length of $1m$; it also determines the constant of dielectric permittivity of free space which is a derived constant as $\epsilon_o = 1/(\mu_o c_o^2) = 8.854187817... \times 10^{-12} F \cdot m^{-1}$.

So far, I have not been clear as to whether we have described our wave equation in terms of time and space or as its Fourier transform in terms of frequency and wavenumber. Let us, then, at place $\vec{r}$, define $\vec{E}(\vec{r}, t) = \vec{\mathrm{E}}(\vec{r}, \omega) e^{-i\omega t}$ where, now, $\vec{\mathrm{E}}(\vec{r}, \omega)$ is the current complex-valued amplitude of the ω-frequency component of the $\vec{E}$-field at place $\vec{r}$. Note the subtle change of font ($\vec{\mathrm{E}}$ vs $\vec{E}$) to indicate the Fourier transformed amplitude. Also note that in reference to the notation used earlier for the Fourier amplitude transformed with respect to both time and place,

$$\vec{\mathrm{E}}(\vec{r}, \omega) = \vec{\mathcal{E}}(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{r}},$$

$$\vec{\mathrm{H}}(\vec{r}, \omega) = \vec{\mathcal{H}}(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{r}}.$$

Substituting into the wave equation in terms of the electric field above,

$$\nabla^2 \vec{\mathrm{E}}(\vec{r}, \omega) + \omega^2 \mu\epsilon \vec{\mathrm{E}}(\vec{r}, \omega) = 0.$$

2.1.2 $\sigma \neq 0$

If we were again to allow for non-zero conductivity in our medium, this equation extends as:

$$\nabla^2 \vec{\mathrm{E}}(\vec{r}, \omega) + (\omega^2 \mu\epsilon - i\omega\sigma)\vec{\mathrm{E}}(\vec{r}, \omega) = 0.$$

Forming $\gamma^2 = i\omega\sigma - \omega^2\mu\epsilon$ and suppressing the position-frequency dependence for tidiness,

$$\nabla^2 \vec{\mathrm{E}} = \gamma^2 \vec{\mathrm{E}}$$

and similarly

$$\nabla^2 \vec{\mathrm{H}} = \gamma^2 \vec{\mathrm{H}}.$$

You might note that ω^2/γ^2 has units of a speed-squared, the speed of the propagation of the EM wave. This measure in complex-valued in presence of a conductive medium.

Now, reconsidering Maxwell's equations, above, namely,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$$

and

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} = \epsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E}$$

in terms of their Fourier amplitudes,

$$\nabla \times \vec{\mathrm{E}} = i\omega\mu\vec{\mathrm{H}}$$

and

$$\nabla \times \vec{\mathrm{H}} = (-i\omega\epsilon + \sigma)\vec{\mathrm{E}}.$$

2.2 Wave impedance tensor

Describing $\vec{\mathrm{E}} : [\,\mathrm{E}_x\, \mathrm{E}_y\, \mathrm{E}_z\,]$ and $\vec{\mathrm{H}} : [\,\mathrm{H}_x\, \mathrm{H}_y\, \mathrm{H}_z\,]$, we can form the *wave impedance tensor*

$$Z_{ij} = \begin{pmatrix} \mathrm{E}_x/\mathrm{H}_x & \mathrm{E}_x/\mathrm{H}_y & \mathrm{E}_x/\mathrm{H}_z \\ \mathrm{E}_y/\mathrm{H}_x & \mathrm{E}_y/\mathrm{H}_y & \mathrm{E}_y/\mathrm{H}_z \\ \mathrm{E}_z/\mathrm{H}_x & \mathrm{E}_z/\mathrm{H}_y & \mathrm{E}_z/\mathrm{H}_z \end{pmatrix}$$

and the *wave-tilt tensors* here, in terms of the **E**-field vector components

$$W_{ij} = \begin{pmatrix} 1 & \mathrm{E}_x/\mathrm{E}_y & \mathrm{E}_x/\mathrm{E}_z \\ \mathrm{E}_y/\mathrm{E}_x & 1 & \mathrm{E}_y/\mathrm{E}_z \\ \mathrm{E}_z/\mathrm{E}_x & \mathrm{E}_z/\mathrm{E}_y & 1 \end{pmatrix}.$$

The various ratios which determine the elements of the wave impedance tensor as measured at the ground surface can be inverted (in simple cases) for subsurface electromagnetic structure[1]. We measure the component $\mathbf{E}_{?}$ and $\mathbf{H}_{?}$ on the surface to determine several elements of the impedance tensor. We can also, similarly, obtain near surface measures of the **E**-field components to obtain the wave-tilt tensor. This can be done using air-borne instrumentation provided we fly closer to the ground than one wavelength equivalent of the electromagnetic wave field above the surface.

We are now prepared with the mathematical formalism to describe a simple magnetotelluric problem. Let assume that we have an electromagnetic wave propagating in the $\boldsymbol{x-z}$-direction intercepting the surface and refracting into the half-space ground as it propagates. Let us presume that we have no variations in the EM-fields or the ground properties with respect to the $\boldsymbol{y}$-direction (that is, in and out of the page in the diagram above). That is all differentials of the forms $\boldsymbol{\partial()/\partial y = \partial^2()/\partial y^2 = 0.}$

2.2.1 H-polarization

We shall deal with a particular polarization of the incident wave (horizontal $\boldsymbol{H}$-polarization: $\boldsymbol{H_x = H_z = 0}$). Then, as the E-field vector is necessarily orthogonal to the wave's propagation direction and to the H-field, $\boldsymbol{E_y = 0.}$

$$\begin{aligned}
\frac{\partial^2 \mathrm{H}_y}{\partial x^2} + \frac{\partial^2 \mathrm{H}_y}{\partial y} &= \gamma^2 \mathrm{H}_y, \\
-\frac{\partial \mathrm{H}_y}{\partial z} &= (-i\omega\epsilon + \sigma)\mathrm{E}_x, \\
\frac{\partial \mathrm{H}_y}{\partial x} &= (-i\omega\epsilon + \sigma)\mathrm{E}_z.
\end{aligned}$$

Note that $\mathbf{H}_{\boldsymbol{y}}$, $\mathbf{E}_{\boldsymbol{x}}$ and $\mathbf{E}_{\boldsymbol{z}}$ can and do vary with respect to place and especially with respect the $\boldsymbol{z}$-direction. Boundary conditions on our free-space to half-space surface determine how.

In this case where variations in the $\boldsymbol{y}$-direction have been suppressed, partial differential equations of the form

$$\left(\nabla^2 - \gamma^2\right)\Psi = 0$$

have *general solutions* of form

$$\Psi = (c_1 e^{-ik_z z} + c_2 e^{+ik_z z})(c_3 e^{-ik_x x} + c_4 e^{+ik_x x})$$

with $\boldsymbol{\gamma^2 = k_x^2 + k_y^2.}$ While it might not be obvious to you, our propagation vector, $\vec{\boldsymbol{k}} : [\,\boldsymbol{k_x}\,\mathbf{0}\,\boldsymbol{k_z}\,]$ determines the direction and wavenumber of our EM-wave. Boundary

[1] NRCan's Magnetic Plotting Service shows such component electric and magnetic field components as measured at several geomagnetic observatories around the country: Geomagnetism Canada

conditions appropriate to our problem bring us to the *particular solutions*. In our diagram, the incoming wave is travelling in $\boldsymbol{x}$-positive direction so that

$$\mathrm{H}_y = (\mathcal{H}_- e^{-ik_z z} + \mathcal{H}_+ e^{ik_z z}) e^{ik_x x}.$$

The equivalent E-field vectors are then simply,

$$\mathrm{E}_x = \frac{ik_z}{-i\omega\epsilon + \sigma} \left(\mathcal{H}_- e^{-ik_z z} - \mathcal{H}_+ e^{ik_z z}\right) e^{ik_x x},$$

$$\mathrm{E}_z = \frac{ik_x}{-i\omega\epsilon + \sigma} \left(\mathcal{H}_- e^{-ik_z z} + \mathcal{H}_+ e^{ik_z z}\right) e^{ik_x x}.$$

2.2.2 Response of a homogeneous halfspace to H_y polarization

We have all the tools, now, to apply boundary conditions to a particular problem. As example, we shall look at the response of a homogeneous halfspace ground to an impinging EM-wave that is horizontally polarized. The wave is propagating in the $\boldsymbol{x}-$direction; it is plane-polarized with its $\vec{H} : [\, 0 \;\; H_y \;\; 0 \,]$.

The condition that must hold at the boundary between the overlying freespace and the lower halfspace reduce to the following ($\hat{n}$ is the unit normal vector on the boundary):

$\hat{n} \cdot (\vec{B}_0 - \vec{B}_1) = 0$, continuity of the magnetic induction field normal to the boundary,
$\hat{n} \times (\vec{E}_0 - \vec{E}_1) = 0$, continuity of the electric field tangential to the boundary,
$\hat{n} \times (\vec{H}_0 - \vec{H}_1) = J_s$, continuity of magnetic field tangential to the boundary with a possible current density J_s, and
$\hat{n} \cdot (\vec{D}_0 - \vec{D}_1) = \rho_s$, continuity of the dielectric field normal to the boundary with possible surface charge ρ_s.

The subscripts "$_0$" and "$_1$" designate the fields in the freespace and ground halfspace respectively.