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# The stochastic excitation of reversals in simple dynamos

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Several variations of the $\alpha\omega$ disc dynamo models are known to exhibit chaotic magnetic field behaviour typical of non-linear, dissipative systems. Although these models demonstrate that geomagnetic reversals can be generated by simplified dynamo equations, the behaviour of the magnetic field itself is generally too simple, showing especially an absence of long polarity epochs in most of the models. We show that the addition of three varieties of stochastic processes (Gaussian, flicker and brown noise) enriches the field evolution and can lead to realistic palaeomagnetic behaviour. We argue that noise processes must be present in the actual fluid core and suggest, from a physical point of view, a flicker noise stimulation of the dynamo. We find two features of the palaeomagnetic record that would favour the presence of noise in the dynamo process, namely the absence of a linear oscillation in field intensity between reversals or, even if present, the absence of an increase in amplitude of this oscillation prior to a reversal. We also consider the addition of a random component to the helicity driving function of the $\alpha^2$ dynamo process and show that various types of reversal can occur. Unfortunately, realistic field behaviour cannot be maintained over long time periods due to the tendency for the magnitudes of the poloidal and toroidal fields to equilibrate during a polarity epoch.

## 1. Introduction

Geomagnetic reversals have recently been attributed to the inherent non-linear behaviour of the equations governing magnetohydrodynamic interactions in the Earth's core (Chilingworth and Holmes, 1980; Krause and Roberts, 1981). Although examination of the full equations seems an impossible task, the simple disc dynamo models provide rewarding insights into the processes responsible for field generation by dynamo action (Bullard, 1955). In particular, it has been shown that reversals of the magnetic field appear on integration of the simple disc dynamo equations and are associated with chaotic behaviour, governed by the appearance of strange attractors in the trajectory space of the equations (e.g., May, 1976). It is natural to consider whether geomagnetic reversals have to be accepted simply as indeterminism in the governing equations; if so, this situation would provide little physical insight into the details of planetary magnetism and little comfort for those wrestling with the interpretation of palaeomagnetic results.

The equations describing simple dynamo models are known to exhibit instabilities with respect to starting conditions and integration method, and it has been shown that in common with other systems (e.g., Sparrow, 1982) the existence of strange attractors and chaotic behaviour is inher-

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ent in the equations. The precise initial conditions, as well as numerical round-off error, are known to affect the details of chaotic solutions and in all physically realistic situations the addition of some noise process (random or otherwise) must intrude to influence the evolution of the system. The role of noise has been considered by Crutchfield and Hubermann (1980) for the one-dimensional anharmonic oscillator and by Guckenheimer (1982) for experimental data in general. For fluid motion in the Earth's core, and other similar problems concerning planetary interiors, irregularities either in material properties, physical or chemical processes or fluid dynamical turbulence will provide an adequate source of noise.

The task then is to examine whether the essential behaviour of simple disc dynamo models can be attributed to inherent chaos or to 'external' stimulation by a random component. This component could be either internal to the core (such as a fluctuation in a local background magnetic field or in convective fluid velocity) or external (such as fluctuations in electromagnetic or mechanical mantle torques or a near surface event such as an earthquake). We begin by reviewing Rikitake's (1958) dynamo and the single disc dynamo of Robbins (1977) and argue that Robbins' dynamo is the most suitable candidate for the introduction of a stochastic component. We show that integration error itself must stimulate reversals in the chaotic regime of the Robbins dynamo. We then stimulate this dynamo with white noise and examine its non-linear response by numerical methods and find that it is resonant at the frequency of the primary oscillation, a result which is not surprising and that probably applies to most disc dynamos. Although this indicates that the nature of the stochastic excitation within the resonant band is irrelevant, general considerations favour a fractal type of stimulation as proposed for the Chandler Wobble (Jensen and Mansinha, 1984). Various other simple dynamo models (Allan, 1962; Krause and Roberts, 1981) are also examined from the same point of view.

Finally we briefly examine the stochastic excitation of the $\alpha^2$ dynamo model proposed by Olson (1983). Owing to different physical assumptions about the field generation process, the response of this model to stimulation through a helicity source term is not masked by the linear oscillation as in the disc dynamo models. Although we find the model to have some undesirable features, such as the requirement for frequent reversals to maintain a necessary asymmetry between the poloidal and toroidal fields, the connection of helicity with thermodynamics in the core gives added interest to this approach.

## 2. Numerical excitation of chaotic dynamos

### *2.1. Rikitake's dynamo*

Rikitake's double disc dynamo model is known to possess self reversals when the equations are integrated numerically (Rikitake, 1958) and Cook and Roberts (1970) demonstrated that the system shows all the attributes of chaotic behaviour associated with non-linear systems. In one interpretation of this model, the magnetic fields crossing the two discs can be thought of as representing the poloidal and toroidal fields of the Earth's core (Krause and Roberts, 1981); alternatively (Bullard, 1955) the model could apply to two large eddies in the outer core. The equations of motion depend on two non-dimensional parameters, $K$ (related to the difference in angular velocities between the two discs) and $\mu$, the ohmic dissipation. The trajectories (coil current plotted against the angular velocity) oscillate from one stable point (normal polarity) to the other (reversed polarity) for all reasonable values of these parameters, although in some parametric regimes the transition is periodic while in others it is irregular.

Subsequently Ito (1980) found for each $K$ value a regime for $\mu$ in which the transitions (reversals) were highly irregular and non-stationary and suggested that the Earth's core would tend to this 'minimum entropy' regime. The stable points of the Rikitake dynamo are, however, repulsive, i.e., all trajectories which start arbitrarily close to the stable points eventually spiral away, for every value of (K, $\mu$), and the dynamo has no parameter regime with a stable state. Therefore this model is not a good candidate for additional stochastic excitation, although later on we will consider a modification of this dynamo, due to Allan (1962), that incorporates viscous damping.

*2.2. Robbins' dynamo*

The Robbins dynamo (Robbins, 1977) consists of the original Bullard single disc, homopolar, dynamo with an impedance between the brush and coil and a shunt connected across the coil. This modified disc dynamo is governed by the following equations

$$\begin{aligned} \dot{w} &= R - zy - \nu w \\ \dot{z} &= wy - z \\ \dot{y} &= \sigma(z - y) \end{aligned} \tag{1}$$

in which $y$ is the current in the coil (velocity of the fluid), $z$ the current in the disc (horizontal temperature distribution) and $w$ the angular velocity of the disc (vertical temperature distribution). The non-dimensional parameters are $R$, the driving torque (mechanical convective force, e.g., temperature gradient), $\nu$, the bearing friction (viscous dissipation) and $\sigma$ (the ratio of disc current decay time to coil current decay time). The variables were interpreted by Robbins according to the thermal convection model of dynamo action and should be re-assigned according to either gravitationally driven convection or to other models as appropriate.

Equations (1) involve three parameters, compared to two in the Rikitake model, and the solutions exhibit different behaviour in different regimes of the parameter $R$. The solutions for increasing values of $R$ are defined below, with the numerical values { } evaluated for $\nu = 1$ $\sigma = 5$, as in Robbins' paper:

*Regime I*: $R < R_0 = \nu$ $\{= 1\}$
all solutions approach the zero field state $(w, z, y) = (R/\nu, 0{,}0)$.

*Regime II*: $R_0 < R < R_{sc}\{= 7.26175\}$
all solutions approach the stable points ('cell' solutions) $(w, z, y) = (1, \pm\sqrt{R-\nu}, \pm\sqrt{R-\nu})$ without $y(t)$ changing sign (no reversal).

*Regime III*: $R_{sc} < R < R_{cc}\{= 14.455\}$
as Regime II but $y(t)$ can change sign (reversal).

*Regime IV*: $R_{cc} < R < R_c = \sigma\nu(\sigma + \nu + 3)/(\sigma - \nu - 1)$ $\{= 15.0\}$
this is the subcritical regime where the number of oscillations between reversals is irregular. Solutions initially close to the stable points eventually decay to the stable points, whereas solutions initially far from the stable points reverse indefinitely.

*Regime V*: $R > R_c$
all solutions, even those beginning at the stable points, become unstable and oscillate indefinitely about the stable points.

In Regime IV, for $R$ only slightly greater than $R_{cc}$, there can be a large number of oscillations between 'reversals' and the reversal sequence appears locally to be highly non-stationary. Because this best mimics the palaeofield behaviour, the Robbins dynamo parameters are assumed to lie in this regime. Unfortunately, unlike the Rikitake dynamo (Ito, 1980), there does not appear to be a 'minimum entropy' condition for this dynamo where one might, a priori, expect to find the parameters.

To illustrate the effect of round-off error on the solutions, we show in Fig. 1 three plots of $y(t)$ for different integration error tolerances. The parameters chosen were $\nu = 1$, $\sigma = 5$ and $R = 13$ for Regime III in which the solutions are expected to eventually decay to the stable points. A fourth order Runge–Kutta method was used with automatic step size determined by the error tolerance, using double precision arithmetic (on an IBM-PC and a COMPAQ portable with 8087 co-processors).

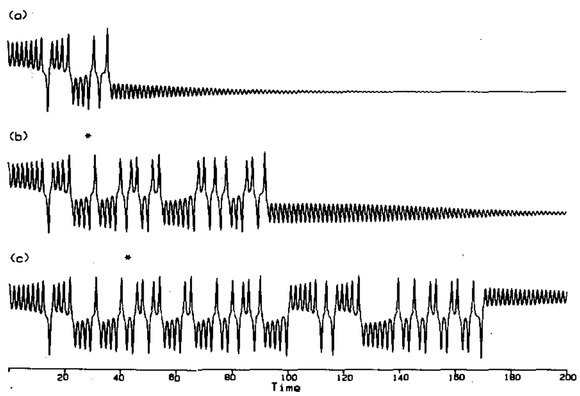



Fig. 1. Plots of $y(t)$ for Robbins' dynamo with $\nu = 1$, $\sigma = 5$, $R = 13$ for three values of error tolerance (a) $10^{-4}$ (b) $10^{-6}$ (c) $10^{-8}$. The asterisks indicate where adjacent plots first differ in time (increasing). The time step for plotting the variables is 0.1 units.

It can be seen that as the error tolerance improves, the trajectory departs after a time (denoted by the asterisks) from the previous plot and the subsequent evolution after this time is quite different. By induction, one can argue that, regardless of the numerical accuracy of any particular integration scheme or the power of any computer, the eventual behaviour of any one realisation (including reversals) of this type of dynamo is governed by numerical inaccuracies. On the other hand, as the initial part of each plot demonstrates, reversals obviously do occur as a result of chaos in the non-linear equations and are caused by more than numerical approximations.

Figure 1 also illustrates other behaviour which we shall note for future reference. In particular, the time prior to a reversal is always preceded by a growing instability of the primary oscillation, as observed by Bullard (1978); we have never seen a reversal of this, or any other unforced, dynamo model that does not show this feature. Secondly, once decaying oscillations begin they continue until the stable solutions are obtained and there is no subsequent reappearance of growth towards a reversal. At the present time we do not know if these conditions are met by the actual geodynamo because of the limited palaeomagnetic record, but in principle these might be testable consequences. Finally, we should note from this and previous work that the concept of stationarity of the reversal sequence is very hard to determine either experimentally or numerically: the former is plagued by sampling and age determination problems, the latter by the critical choice of parameters in the simplified models.

## 3. Stochastic excitation of the driving torque

We now introduce an 'external' stochastic component into the equations of motion to simulate what we assume must be irregularities in the underlying hydrodynamics of the Earth's core. Under some form of core convection, especially that due to gravitational settling (e.g., Loper, 1978; Gubbins and Masters, 1979), one may readily imagine clumps of denser or lighter fluid forming locally and giving rise to small-scale irregularities in a convective velocity field. This component would, through the Lorentz force in the Navier–Stokes equation, generate local irregularities in the magnetic field as well. Alternatively, perhaps the core conductivity, due to compositional variations, can vary locally thus affecting the balance of diffusive to generated magnetic flux in the induction equation. Perhaps instead there is a change of conditions at the core–mantle boundary due to earthquakes or mantle torques. We shall lump all these effects into a single driving term which has pre-determined statistics and is inserted into the equations of motion.

### 3.1. Robbins' dynamo

For the Robbins dynamo, we modify eq. 1 by allowing

$$R = \bar{R} + r(t) \tag{2}$$

where we consider $r(t)$ to be one of the three processes illustrated in Fig. 2. To begin with we choose a pseudo-random uncorrelated Gaussian process ('white noise') which is stationary with zero mean and specified variance $\sigma_1^2$. This is injected into the dynamo eq. 1 at uniform time intervals $dt$ (not necessarily coincident with the elementary integration steps). Because of the approximately uniform sample spectrum of the noise process (the true power spectrum is equal to the variance of the noise), the amount of power delivered to the dynamo by this process is linearly proportional to the interval $dt$.

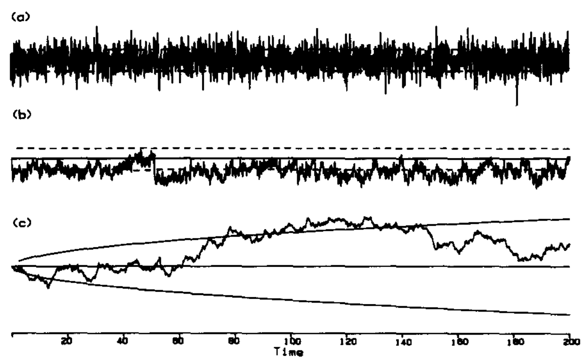
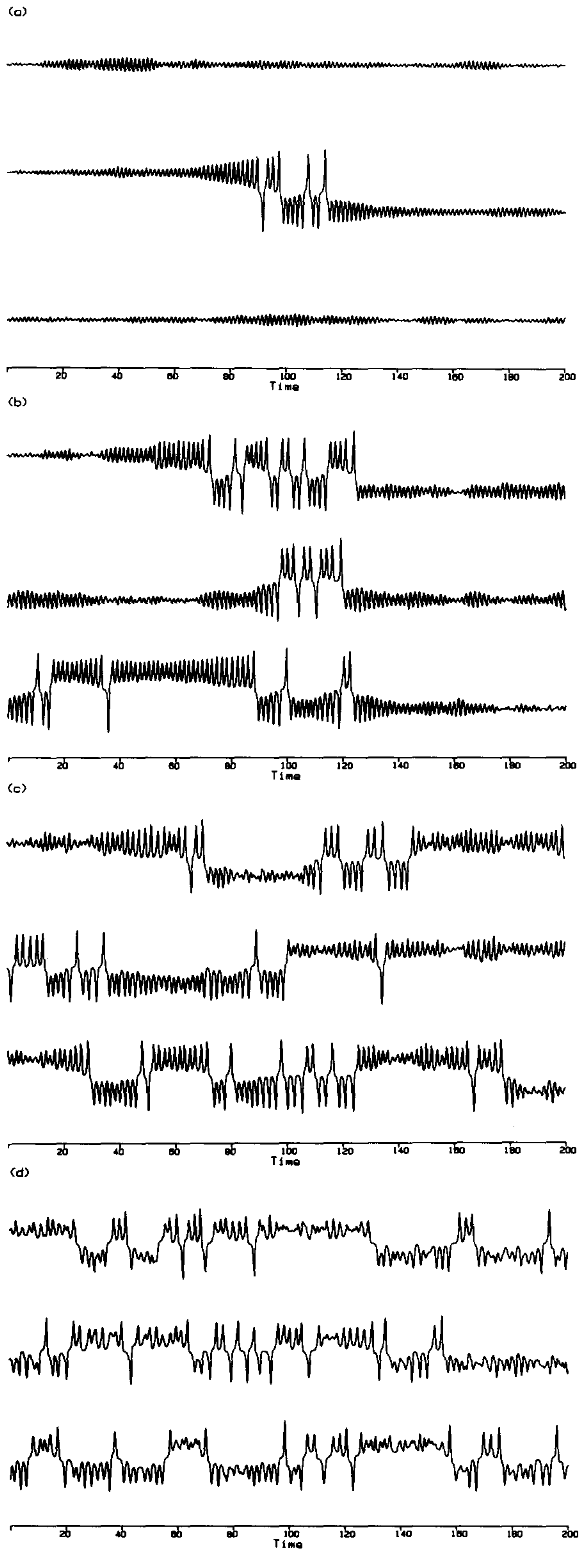



Fig. 2. Three random processes (a) Gaussian pseudo-white noise (b) flicker noise (c) brown noise (random walk) each with 200 samples per division ($dt = 0.1$). The uniform random number generator was re-seeded at the beginning of each trace to a value of 0.1. The flicker noise sequence repeats after $2^{24}$ samples or about 8400 lines like (b).




In Fig. 3 we show four examples of the dynamo in Regime III for different combinations of $\bar{R}$ and $\sigma_1$. The integrations are begun at the stable points where they would remain throughout in the absence of excitation. One can see in Fig. 3(a) a long sequence of one polarity with varying amounts of stimulation, including growing and decaying fields without reversals, followed by a burst of reversal activity in the middle of the sequence. Figure 3(b) demonstrates the effect of decreasing $\bar{R}$ and increasing the excitation, and shows how isolated polarity excursions can occur both with and without long term changes of polarity. Again, however, as for the chaotic regime $V$ the reversals are initiated by growing instability in the primary oscillation.

More extreme examples of excitation are shown in Fig. 3(c) and (d) where we have decreased $\bar{R}$ and increased $\sigma_1$ (in (2)) to maintain a value of $\bar{R}+\sigma_1=R_c$. One can see that as $\bar{R}$ is decreased the oscillations and reversals become more irregular and more frequent and in particular the behaviour prior to a reversal loses the characteristic of a gradual build-up of amplitude. At the level of Fig. 3(d), the random part of $R$ is almost 100% of the mean, which may or may not be a drastic fluctuation in driving torque for the geodynamo (depending on one's notion of how important turbulent behaviour may be). Nonetheless, it is clear that with the addition of white noise excitation, the Robbins dynamo now exhibits a wider range of behaviour than a purely chaotic dynamo and with a much greater range of R than previously assumed. Other tests have shown that for very modest levels of noise excitation the trajectories are indistinguishable from those in a purely chaotic dynamo, as expected.

Owing to the fast oscillation in the Robbins dynamo, even for strongly random excitation, we have investigated the non-linear response of the dynamo using sinusoidal excitation, $R=\bar{R}+A\sin\omega t$, similar to the periodic forcing considered by Chilingworth and Holmes (1980). An ini-

Fig. 3. Plots of $y(t)$ for Robbins' dynamo (a) $\bar{R}=14$, $\sigma_1=1$ (b) $\bar{R}=13$, $\sigma_1=2$ (c) $\bar{R}=11$, $\sigma_1=4$ (d) $\bar{R}=8$, $\sigma_1=7$; error tolerance of $10^{-5}$, other parameters as Fig. 1. The times scale applies to only the first trace of each plot, for subsequent traces add 200 per line.

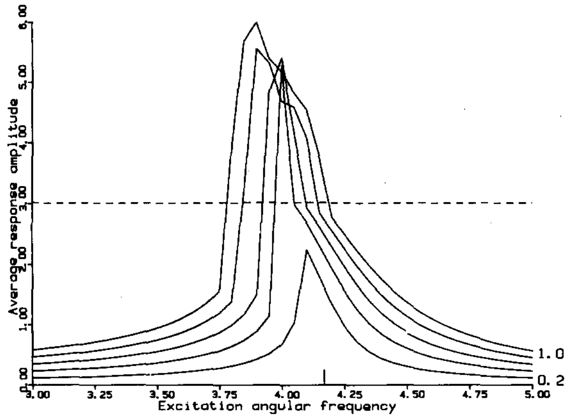



Fig. 4. Response (mean deviation of $y(t)$ from stable point) of the Robbins' dynamo to sinusoidal excitation for five values of $A$ (0.2 to 1.0); other values as for Fig. 3(b). The vertical line at 4.17 rad is the theoretical linearised resonance frequency. The horizontal dashed line shows the approximate response amplitude for the onset of reversals; amplitudes beyond this value are less dependable than those below it.

tial trial with a sweep sinusoid indicated that the fast oscillation was sensitive to angular frequencies in the range 3–5 rad per unit time and a detailed numerical search was made of the dynamo response (measured by the mean deviation of $y(t)$ from its initial value achieved in 200 time units) for various values of $A$ for $\overline{R} = 13$ (Fig. 4). The theoretical linearised response can be obtained from (1) by simple perturbation theory, giving the resonant frequencies as the roots of the characteristic equation

$$\lambda^3 + (\sigma + \nu + 1)\lambda^2 + (R + \sigma\nu)\lambda + 2\sigma(R - \nu) = 0 \tag{3}$$

For the values in Fig. 3(a) we find that the imaginary part of $\lambda = 4.17$ which, as indicated in Fig. 4, is in fair agreement with the peak of the response of the lowest (0.2) amplitude value for $A$. As is typical of non-linear systems, the resonant frequency is amplitude dependent and in this case moves to lower values as the excitation increases. The Robbins dynamo is clearly responsive to frequencies only in a narrow range about the value 4.0 and this accounts for the persistence of the fast (linear) oscillation in all the trajectories. Although we are reluctant to interpret the time axis of Fig. 3 (and the other similar plots in this paper) geophysically, if one believes the dynamo has amplitude fluctuations of ~ 8000 y (McFadden, 1984), then this period might be assigned to the linear oscillation. Each division of the time axis (20 units) would then be approximately 13 cycles or 100 000 y so each line would be about 1 Ma.

One of the features of white noise excitation is that it is uncorrelated. This can lead to unphysical requirements in the dynamical system, especially when there are very large and rapid excursions in driving torque. An alternative model for the noise inherent in physical systems, which has become fashionable due to the work of Mandlebrot (1983), is that of self-similar or 'fractal', 'one-over-f' noise. Here, we shall call this stochastic process 'flicker noise' because it is analogous to the phenomenon of that name in solid-state electronic systems which dominates all other sources of noise at very low frequencies. Flicker noise is correlated over all times with a magnitude that decreases as the correlation interval increases such that the process ideally possesses a $1/f$ power spectrum. Flicker noise is properly stationary if low-frequency limited; it is, in fact, that self-similar or fractal random sequence which possesses the highest degree of self-correlation while being stationary.

Physically, flicker noise allows us to describe a random dynamo which has the longest possible memory and the widest possible spatial correlation while not evolving through geological time. That is, we shall demand that those parameters which define the deterministic component of the dynamo and the statistical measures of the randomness inherent in the dynamo (the variance and correlation measures) remain fixed over the duration of our simulation or observation. Though the search for memory in the palaeomagnetic behaviour (magnetic field components) has been elusive (McFadden, 1984), temporal correlation as a property of an excitation process introduces a more subtle memory characteristic.

To demonstrate the effect of self-correlation in the excitation of a driven dynamo, we show three examples of Robbins' dynamo forced by flicker noise (Fig. 5). Before discussing the results of these simulations, however, it is worthwhile to note the level of excitation required in comparison to that of white noise. The theoretical power spectra of frequency band-limited white noise, $P_1(f)$, and of flicker noise, $P_2(f)$, are given by

$$P_1(f) = \sigma_1^2/2f_N$$
$$P_2(f) = \sigma_2^2/2\ \ln(f_N/f_L)\,|f| \qquad (4)$$

where $\sigma_1^2$, and $\sigma_2^2$ are the variances of the two processes. The sharp lower, $f_L$, and upper, $f_N$, band-width cutoff frequencies are determined in these simulations by

$$f_N = 1/2dt$$
$$f_L = 1/T \qquad (5)$$

where $dt$ is the temporal sampling interval and $T$ the duration of the repetition cycle of the simulated flicker noise. For the simulations presented here, $T \approx 1.7 \times 10^7\ dt$. To balance the flicker power density at any frequency $f$ with the equivalent white power density at the same frequency, we require that

$$\sigma_2/\sigma_1 = \left[(f/f_N)\ \ln(f_N/f_L)\right]^{1/2} \qquad (6)$$

where, for $f = 0.66$ (the fast oscillation) and $dt = 0.1$, we find $\sigma_2/\sigma_1 \approx 1.5$. Thus we use a standard deviation $\sigma_2 = 3.0$ for $\bar{R} = 13$ to compare with Fig. 3(b) for three different seed values of the random number generator.

It can be seen that the first trace in Fig. 5 (using the flicker noise sequence in Fig. 2) yields no reversals due to the level of excitation being generally lower than the mean. The second trace shows many reversals from a flicker noise sequence (not shown) that is persistently above the mean, while the third trace is somewhere between the other two in response. If this model were correct, one might expect to find weak evidence of correlation in the reversal record, decreasing with the length of time considered. It is also clear that despite the seeming length of the numerical 'polarity epochs', this noise process is stationary with zero bias, a statistical property of the palaeomagnetic record that has been debated for some time. On the basis of these disc dynamo models, however, we have to admit that the dominance of the fast oscillation clearly masks the distinction between the noise processes discussed so far.

Finally, an example of the Robbins dynamo with 'brown noise' excitation (Fig. 2) is shown in Fig. 6. This type of process, known also as a random walk, differs from flicker noise in being non-stationary (as well as highly correlated) and provides magnetic field response similar to the flicker noise process. Owing to the very long correlation lengths possible, the excitation could wander quite far from the mean ($\bar{R}$) for considerable periods of time, thus emphasising any tendency for the reversals to appears as clusters in time. In retrospect, Robbins' dynamo is an ideal model for stochastic excitation because of the transition from stable to unstable regimes in the forcing parameter $R$ in the governing equations.

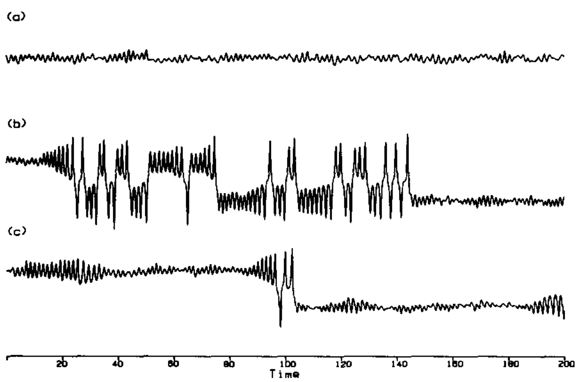



Fig. 5. Plot of $y(t)$ for Robbins' dynamo with flicker noise (a) seed = 0.1 (b) seed = 0.2 (c) seed = 0.3 for $\bar{R} = 13$, $\sigma_1 = 3$. The traces may be regarded as part of the same magnetic field sequence at different 'epochs'.

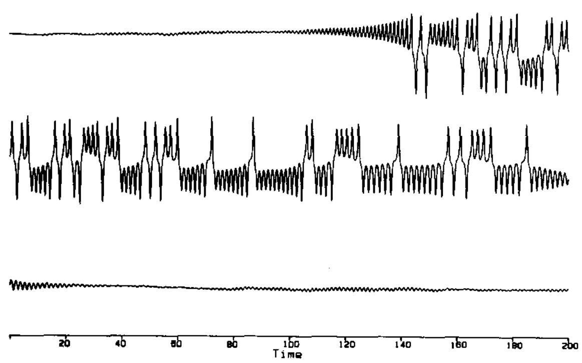



Fig. 6. Plot of $y(t)$ for Robbins' dynamo with the brown noise excitation beginning as in Fig. 2. Unlike Fig. 5, the three traces form a continuous sequence.

### 3.2. *Allan's dynamo*

We have modified Rikitake's dynamo by the device proposed by Allan (1962) of introducing viscous damping into the equation governing the rotation rate. After many trials, we found that the interplay of excitation and damping does not yield such a satisfactory result as for Robbins' dynamo because the parameters have to be very finely balanced to produce a result which is noticeably different from the unforced Rikitake dynamo.

### 3.3. *Krause and Roberts' dynamo*

The simplified dynamo presented by Krause and Roberts (1981) was derived directly from the basic equations of magnetohydrodynamics rather than as a derivative of the homopolar models. Quite remarkably, by reducing the number of free parameters to a minimum (i.e., one), the equations of motion obtained by Krause and Roberts

$$\begin{aligned} \dot{P} &= T - P \\ \dot{T} &= \Omega P - T \\ \dot{\Omega} &= \kappa^2(1 - PT) - \beta\Omega \end{aligned} \tag{7}$$

are almost identical to Robbins' dynamo (1) with $\sigma = 1$ and $(P, T, \Omega)$ replacing $(y, z, w)$. $P$ and $T$ are interpreted to be the poloidal and toroidal components of the main field, respectively. The only significant difference is the driving term $\kappa^2$ in the third equation of (7) which in Robbins dynamo does not modify the product $PT$. Note we have added a viscous dissipation term in the third equation of (7) to bring it in line with Robbins' and Allan's dynamos, as well as at the suggestion of Krause and Roberts themselves. The single parameter

$$\kappa^2 = \frac{\tau_{mag}}{\tau_{rot}} \tag{8}$$

where $\tau_{mag}$ is the decay time of the magnetic field, and $\tau_{rot}$ is the regeneration time for the differential rotation, governs the amount of mechanical forcing. Krause and Roberts (1981) commented that for the values $\kappa^2 = 0.1$ and $\kappa^2 = 1.0$ this model gives overly simple field behaviour, in particular allowing too little time between reversals compared to the long polarity epochs of the palaeomagnetic field. It has been found by experimentation, however, that the above values chosen for $\kappa^2$ do not typify all the interesting behaviour of this model. Figure 7 shows the situation for $\kappa^2 = 4$ for (a) no excitation (b) excitation but no damping and (c) excitation and damping. The results for the unforced behaviour are, not unexpectedly, strongly reminiscent of Robbins' dynamo. As for Allan's dynamo, while its behaviour can be made more 'erratic' by stochastic forcing, the results are very dependent on the choice of parameters.

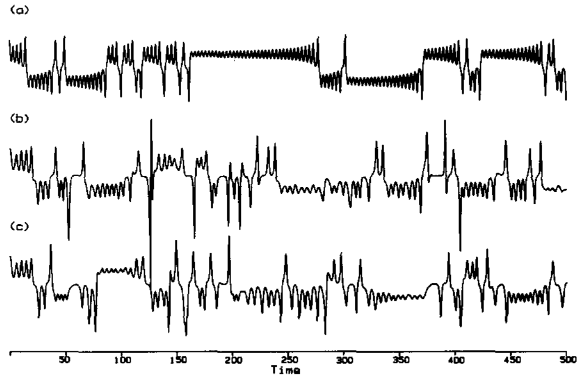



Fig. 7. Krause and Roberts' dynamo for $\kappa^2 = 4$ with (a) $\sigma_2 = \beta = 0$ (b) $\sigma_2 = 2.0$, $\beta = 0$ (c) $\sigma_2 = 2.0$, $\beta = 0.02$.

## 4. Stochastic excitation of Olson's $\alpha^2$ dynamo

In contrast to the previous models which have been $\alpha\omega$ dynamos, we now consider an example of the $\alpha^2$ class of dynamos, in particular the model proposed by Olson (1983). In this model the magnetic field evolution is again derived from the induction equation, but with an alpha factor controlling the generation of the Lorentz force from the main field components. In non-dimensional variables the resulting equations are

$$\begin{aligned} \dot{P} &= -P - \alpha T \\ \dot{T} &= -\alpha P - T \\ \alpha &= \left[2/(P^2 + T^2)\right]\Gamma \end{aligned} \tag{9}$$

where $P$ and $T$ are the main poloidal and toroidal fields and $\alpha(t)$ is the field regeneration parameter. The third equation is the important one in which the $\alpha$ effect is derived from $\Gamma(t)$, a source term representing the helicity of the velocity field. Olson (1983) discussed the evolution of (9) when $\Gamma(t)$ is given simple forms such as a permanent or a transient change from $+1$ to $-1$ with the initial conditions

$$P(0) = 1$$
$$T(0) = -(1+\delta) \qquad (10)$$

where $\delta$ is the initial toroidal field anomaly. Olson (1983) showed (a) after a sign change in $\Gamma(t)$ there can be a change of sign either in $P$ or $T$ separately (component reversal) or together (full reversal) or a fluctuation in amplitude of $P$ or $T$ (excursion) and (b) that the time scales for these changes seems consistent with the observations (though considerably shorter than the free ohmic decay time). Further, Olson speculated that changes in the sign of $\Gamma$, the helicity, may be caused by imbalances in two competing energy sources in the core, fluid turbulence associated with heat loss at the core–mantle boundary and crystallisation at the inner core boundary.

We now consider the consequences of taking Olson's (1983) model one stage further by replacing $\Gamma(t)$ by a stochastic process which might trigger the reversals. Unlike Robbins' dynamo, in which the driving torque could be replaced by a steady part and a random part (eq. 2), reversals in Olson's model depend on $\Gamma(t)$ changing sign, so the steady part of the helicity may be set to zero. As we shall see, this has the unfortunate consequence that the steady solutions of (9), i.e.

$$\alpha_0 = +1 : P_0 = -T_0 = \sqrt{\Gamma}$$
$$\alpha_0 = -1 : P_0 = \quad T_0 = \sqrt{-\Gamma} \qquad (11)$$

are considerably affected by the magnitude of $\Gamma(t)$, so that fluctuations in helicity do not produce a 'two-state' mode of operation as is the case for fluctuations in mechanical torque in the disc dynamos. A straightforward stability analysis of (9) for constant $\Gamma$ confirms that the values (11) are uniformly stable with no linear oscillation and small departures of $P$, $T$ from (11) decay as $e^{-2t}$, independent of $\Gamma$. The response of this dynamo is therefore more dependent on the forcing function than was found for the chaotic dynamos.

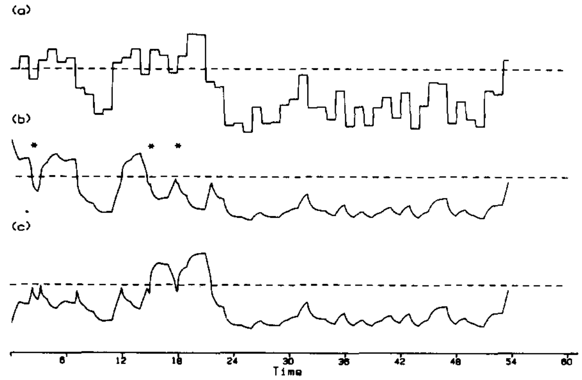



Fig. 8. Plot of (b) $P(t)$ and (c) $T(t)$ for Olson's dynamo with $\Gamma(t)$ given by a flicker noise sequence (a) with standard deviation 1.0, seed 0.3 (cf. Fig. 2). Trace (b) shows a component reversal, a full reversal and an excursion at the asterisks, respectively. Integration tolerance $10^{-5}$, time step 0.1, flicker noise added every 1.0 time units, anomaly $\delta = 0.02$. One time unit is approximately 7300 y.

As an example of the above, we show in Fig. 8 the integration of (9) and (10) when $\Gamma(t)$ is a flicker noise sample, chosen to have several zero crossings. The response of the $P$, $T$ fields are initially encouraging in that, although one field component or the other more or less follows the excitation in sign and magnitude, there are component reversals, full reversals and excursions exactly of the morphology described by Olson. Rather surprisingly though, the model breaks down at time 53.5 due to numerical problems. Closer inspection shows that two factors cause this to happen. The first is that the anomaly $\delta$ in (10), which represents some initial imbalance between the two field strengths, quickly disappears as the record progresses so that after the last reversal at time 21.5 (Fig. 8) the two fields evolve in parallel with the magnitudes of $P$ and $T$ becoming closer. The second, related, problem is that when either $P$ or $T$ goes through zero (at a reversal), and they have nearly identical magnitudes, $\alpha$ approaches infinity and the model breaks down. This behaviour is consistent with physical intuition that the poloidal and toroidal fields should not both be zero at a geomagnetic reversal.

Further insight into this behaviour can be found from the simple step models of $\Gamma(t)$ presented by Olson. Though he discussed the need for a non-zero anomaly, in the examples presented the integrations were never carried far enough to demonstrate the breakdown, as he was mostly concerned with the field behaviour at the first reversal. We have examined a simple repetitive telegraph signal $\Gamma(t)$ which consists of the value $+1$ for time $t_0$ followed by $a-1$ for a (dimensionless) time 1.0. As $t_0$ increases the anomaly decays until a reversal where a small anomaly can be generated by the numerical integration scheme as the reversal occurs. On our computer, when $t_0 > 5$, the anomaly $|P|-|T|$ decays into the numerical round-off ($\sim 10^{-16}$). Therefore all integrations in which the field components remain of one polarity for longer than 5 time units inevitably follow the same fate as in Fig. 8, regardless of the type of helicity function, stochastic or otherwise. Therefore this dynamo model cannot represent, in the form presented by Olson, a description of the field over long periods of time. Using Olson's estimate of the time scales, 5 units here is equivalent to approximately 36 000 y.

One way to remedy the situation is to inject an asymmetry between the $P$ and $T$ fields in (9) so that at a reversal the weaker field can reverse while the stronger does not. We shall not pursue this issue as ideally such an asymmetry should be present in the basic equations, rather than being added arbitrarily to (9) as a numerical afterthought. Nevertheless, our experience with Olson's model suggests that realistic palaeomagnetic models may be found that have a strong physical connection with core processes.

## 5. Discussion

One cannot, on the basis of the examples shown here, decide a priori whether the palaeomagnetic results favour a chaotic dynamo or a 'noisy dynamo', or indeed some combination of the two. It is evident, however, that our model is reminiscent of the empirical models of Cox (e.g., Cox, 1981) in that we also propose a physically-derived stochastic process to trigger the reversals. We also face the same difficulties in proposing realistic tests as to whether our model is consistent with observation. Until the uncertainties due to palaeomagnetic sampling and age determination are reduced, the statistics of the reversals will remain vague enough to allow several possibilities, as at present. However, some advantage has been gained by demonstrating that a stochastic excitation can either add to or replace the excursions of the magnetic field attributed to non-linear behaviour of the equations of motion. Additionally we have provided a simple mathematical mechanism for giving a complicated polarity record while retaining the attractive simplicity of the disc dynamo models.

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